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ADAPTIVE CONTROL

Stability, Convergence, and Robustness

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To our parents

CONTENTS

PREFACE

The objective of this book is to give, in a concise and unified fashion, the major results, techniques of analysis and new directions of research in adaptive systems. Such a treatment is particularly timely, given the rapid advances in microprocessor and multi-processor technology which make it possible to implement the fairly complicated nonlinear and time varying control laws associated with adaptive control. Indeed, limitations to future growth can hardly be expected to be computational, but rather from a lack of a fundamental understanding of the methodologies for the design, evaluation and testing of the algorithms. Our objective has been to give a clear, conceptual presentation of adaptive methods, to enable a critical evaluation of these techniques and suggest avenues of further development.

Adaptive control has been the subject of active research for over three decades now. There have been many theoretical successes, including the development of rigorous proofs of stability and an understanding of the dynamical properties of adaptive schemes. Several successful applications have been reported and the last ten years have seen an impressive growth in the availability of commercial adaptive controllers.

In this book, we present the *deterministic* theory of identification and adaptive control. For the most part the focus is on linear, continuous time, single-input single-output systems. The presentation includes the algorithms, their dynamical properties and tools for analysis—including the recently introduced averaging techniques. Current research in the adaptive control of multi-input, multi-output linear systems and a class of nonlinear systems is also covered. Although continuous time algorithms occupy the bulk of our interest, they are presented in such a way as to enable their transcription to the discrete time case.

A brief outline of the book is as follows: Chapter 0 is a brief historical overview of adaptive control and identification, and an introduction to various approaches. Chapter 1 is a chapter of mathematical preliminaries containing most of the key stability results used later in the book. In Chapter 2, we develop several adaptive identification algorithms along with their stability and convergence properties. Chapter 3 is a corresponding development for model reference adaptive control. In Chapter 4, we give a self contained presentation of averaging techniques and we analyze the rates of convergence of the schemes of Chapters 2 and 3. Chapter 5 deals with robustness properties of the adaptive schemes, how to analyze their potential instability using averaging techniques and how to make the schemes more robust. Chapter 6 covers some advanced topics: the use of prior information in adaptive identification schemes, indirect adaptive control as an extension of robust non-adaptive control and multivariable adaptive control. Chapter 7 gives a brief introduction to the control of a class of nonlinear systems, explicitly linearizable by state feedback and their adaptive control using the techniques of Chapter 3. Chapter 8 concludes with some of our suggestions about the areas of future exploration.

This book is intended to introduce researchers and practitioners to the current theory of adaptive control. We have used the book as a text several times for a one-semester graduate course at the University of California at Berkeley and at Carnegie-Mellon University. Some background in basic control systems and in linear systems theory at the graduate level is assumed. Background in stability theory for nonlinear systems is desirable, but the presentation is mostly self-contained.

Acknowledgments

It is a pleasure to acknowledge the contributions of the people who helped us in the writing of this book. We are especially appreciative of the detailed and thoughtful reviews given by Charles Desoer of the original Ph.D. dissertation of the second author on which this book is based. His advice and support from the beginning are gratefully acknowledged. Brian Anderson and Petar Kokotovic offered excellent critical comments that were extremely helpful both in our research, and in the revisions of the manuscript. We also thank Karl Astrom and Steve Morse for their enthusiasm about adaptive control, and for fruitful interactions.

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Shankar Sastry
Marc Bodson

Berkeley, California

CHAPTER 0
INTRODUCTION

0.1 IDENTIFICATION AND ADAPTIVE CONTROL

Most current techniques for designing control systems are based on a good understanding of the plant under study and its environment. However, in a number of instances, the plant to be controlled is too complex and the basic physical processes in it are not fully understood. Control design techniques then need to be augmented with an identification technique aimed at obtaining a progressively better understanding of the plant to be controlled. It is thus intuitive to aggregate system identification and control. Often, the two steps will be taken separately. If the system identification is recursive—that is the plant model is periodically updated on the basis of previous estimates and new data—identification and control may be performed concurrently. We will see *adaptive control*, pragmatically, as *a direct aggregation of a (non-adaptive) control methodology with some form of recursive system identification.*

Abstractly, system identification could be aimed at determining if the plant to be controlled is linear or nonlinear, finite or infinite dimensional, and has continuous or discrete event dynamics. Here we will restrict our attention to finite dimensional, single-input single-output linear plants, and some classes of multivariable and nonlinear plants. Then, the primary step of system identification (structural identification) has already been taken, and only parameters of a fixed type of model need to be determined. Implicitly, we will thus be limiting ourselves to *parametric system identification*, and *parametric adaptive control.*

Applications of such systems arise in several contexts: advanced flight control systems for aircraft or spacecraft, robot manipulators, process control, power systems, and others.

Adaptive control, then, is a technique of applying some system identification technique to obtain a model of the process and its environment from input-output experiments and using this model to design a controller. The parameters of the controller are adjusted during the operation of the plant as the amount of data available for plant identification increases. For a number of simple PID (proportional + integral + derivative) controllers in process control, this is often done manually. However, when the number of parameters is larger than three or four, and they vary with time, automatic adjustment is needed. The design techniques for adaptive systems are studied and analyzed in theory for *unknown* but *fixed* (that is, time invariant) plants. In practice, they are applied to *slowly time-varying* and *unknown* plants.

Overview of the Literature

Research in adaptive control has a long and vigorous history. In the 1950s, it was motivated by the problem of designing autopilots for aircraft operating at a wide range of speeds and altitudes. While the object of a good fixed-gain controller was to build an autopilot which was insensitive to these (large) parameter variations, it was frequently observed that a single constant gain controller would not suffice. Consequently, gain scheduling based on some auxiliary measurements of airspeed was adopted. With this scheme in place several rudimentary model reference schemes were also attempted—the goal in this scheme was to build a self-adjusting controller which yielded a closed loop transfer function matching a prescribed reference model. Several schemes of self-adjustment of the controller parameters were proposed, such as the sensitivity rules and the so-called M.I.T. rule, and were verified to perform well under certain conditions. Finally, Kalman [1958] put on a firm analytical footing the concept of a general self-tuning controller with explicit identification of the parameters of a linear, single-input, single-output plant and the usage of these parameter estimates to update an optimal linear quadratic controller.

The 1960s marked an important time in the development of control theory and adaptive control in particular. Lyapunov's stability theory was firmly established as a tool for proving convergence in adaptive control schemes. Stochastic control made giant strides with the understanding of dynamic programming, due to Bellman and others. Learning schemes proposed by Tsypkin, Feldbaum and others (see Tsypkin [1971] and [1973]) were shown to have roots in a single unified framework of recursive equations. System identification (off-line) was

thoroughly researched and understood. Further, Parks [1966] found a way of redesigning the update laws proposed in the 1950s for model reference schemes so as to be able to prove convergence of his controller.

In the 1970s, owing to the culmination of determined efforts by several teams of researchers, complete proofs of stability for several adaptive schemes appeared. State space (Lyapunov based) proofs of stability for model reference adaptive schemes appeared in the work of Narendra, Lin, & Valavani [1980] and Morse [1980]. In the late 1970s, input output (Popov hyperstability based) proofs appeared in Egardt [1979] and Landau [1979]. Stability proofs in the discrete time deterministic and stochastic case (due to Goodwin, Ramadge, & Caines [1980]) also appeared at this time, and are contained in the textbook by Goodwin & Sin [1984]. Thus, this period was marked by the culmination of the analytical efforts of the past twenty years.

Given the firm, analytical footing of the work to this point, the 1980s have proven to be a time of critical examination and evaluation of the accomplishments to date. It was first pointed out by Rohrs and coworkers [1982] that the assumptions under which stability of adaptive schemes had been proven were very sensitive to the presence of unmodeled dynamics, typically high-frequency parasitic modes that were neglected to limit the complexity of the controller. This sparked a flood of research into the robustness of adaptive algorithms: a re-examination of whether or not adaptive controllers were at least as good as fixed gain controllers, the development of tools for the analysis of the transient behavior of the adaptive algorithms and attempts at implementing the algorithms on practical systems (reactors, robot manipulators, and ship steering systems to mention only a few). The implementation of the complicated nonlinear laws inherent in adaptive control has been greatly facilitated by the boom in microelectronics and today, one can talk in terms of custom adaptive controller chips. All this flood of research and development is bearing fruit and the industrial use of adaptive control is growing.

Adaptive control has a rich and varied literature and it is impossible to do justice to all the manifold publications on the subject. It is a tribute to the vitality of the field that there are a large number of fairly recent books and monographs. Some recent books on recursive estimation, which is an important part of adaptive control are by Eykhoff [1974], Goodwin & Payne [1977], Ljung & Soderstrom [1983] and Ljung [1987]. Recent books dealing with the theory of adaptive control are by Landau [1979], Egardt [1979], Ioannou & Kokotovic [1984], Goodwin & Sin [1984], Anderson, Bitmead, Johnson, Kokotovic, Kosut, Mareels, Praly, & Riedle [1986], Kumar and Varaiya [1986], Polderman [1988] and Caines [1988]. An attempt to link the signal processing viewpoint with the adaptive control viewpoint is made in Johnson [1988]. Surveys of the applications of adaptive control are given in a book by Harris & Billings [1981], and in books edited by Narendra & Monopoli [1980] and Unbehauen [1980]. As of the writing of this book, two other books on adaptive control by Astrom & Wittenmark and Narendra & Annaswamy are also nearing completion.

In spite of the great wealth of literature, we feel that there is a need for a "toolkit" of methods of analysis comparable to non-adaptive linear time invariant systems. Further, many of the existing results concern either algorithms, structures or specific applications, and a great deal more needs to be understood about the dynamic behavior of adaptive systems. This, we believe, has limited practical applications more than it should have. Consequently, our objective in this book is to address fundamental issues of stability, convergence and robustness. Also, we hope to communicate our excitement about the problems and potential of adaptive control. In the remainder of the introduction, we will review some common approaches to adaptive control systems and introduce the basic issues studied in this book with a simple example.

0.2 APPROACHES TO ADAPTIVE CONTROL

0.2.1 Gain Scheduling

One of the earliest and most intuitive approaches to adaptive control is gain scheduling. It was introduced in particular in the context of flight control systems in the 1950s and 1960s. The idea is to find auxiliary process variables (other than the plant outputs used for feedback) that correlate well with the changes in process dynamics. It is then possible to compensate for plant parameter variations by changing the parameters of the regulator as functions of the auxiliary variables. This is illustrated in Figure 0.1.

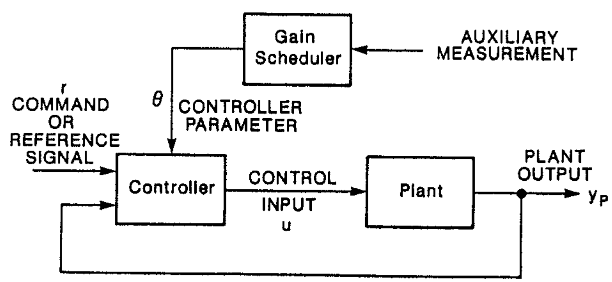

Figure 0.1: Gain Scheduling Controller

The advantage of gain scheduling is that the parameters can be changed quickly (as quickly as the auxiliary measurement) in response to changes in the plant dynamics. It is convenient especially if the plant dynamics depend in a well-known fashion on a relatively few easily measurable variables. In the example of flight control systems, the dynamics depend in relatively simple fashion on the readily available dynamic pressure—that is the product of the air density and the relative velocity of the aircraft squared.

Although gain scheduling is extremely popular in practice, the disadvantage of gain scheduling is that it is an *open-loop* adaptation scheme, with no real "learning" or intelligence. Further, the extent of design required for its implementation can be enormous, as was illustrated by the flight control system implemented on a CH-47 helicopter. The flight envelope of the helicopter was divided into *ninety* flight conditions corresponding to thirty discretized horizontal flight velocities and three vertical velocities. Ninety controllers were designed, corresponding to each flight condition, and a linear interpolation between these controllers (linear in the horizontal and vertical flight velocities) was programmed onto a flight computer. Airspeed sensors modified the control scheme of the helicopter in flight, and the effectiveness of the design was corroborated by simulation.

0.2.2 Model Reference Adaptive Systems

Again in the context of flight control systems, two adaptive control schemes other than gain scheduling were proposed to compensate for changes in aircraft dynamics: a series, high-gain scheme, and a parallel scheme.

Series High-Gain Scheme

Figure 0.2 shows a schematic of the series high-gain scheme.

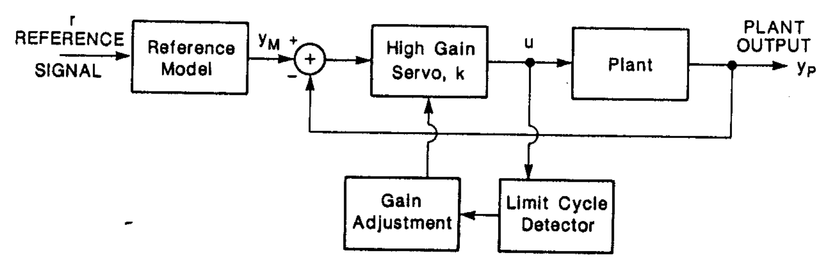

Figure 0.2: Model Reference Adaptive Control—Series, High-Gain Scheme

The reference model represents a pilot's desired command-response characteristic. It is thus desired that the aircraft response, that is, the output y_p, matches the output of the reference model, that is, y_m.

The simple analysis that goes into the scheme is as follows: consider $\hat{P}(s)$ to be the transfer function of the linear, time invariant plant and k the constant gain of the servo. The transfer function from y_m to y_p is $k\hat{P}(s)/1+k\hat{P}(s)$. When the gain k is sufficiently large, the transfer function is approximately 1 over the frequencies of interest, so that $y_m \sim y_p$.

The aim of the scheme is to let the gain k be as high as possible, so that the closed-loop transfer function becomes close to 1, until the onset of instability (a limit cycle) is detected. If the limit cycle oscillations exceed some level, the gain is decreased. Below this level, the gain is increased. The limit cycle detector is typically just a rectifier and low-pass filter.

The series high-gain scheme is intuitive and simple: only one parameter is updated. However, it has the following problems

a) Oscillations are constantly present in the system.
b) Noise in the frequency band of the limit cycle detector causes the gain to decrease well below the critical value.
c) Reference inputs may cause saturation due to the high-gain.
d) Saturation may mask limit cycle oscillations, allowing the gain to increase above the critical value, and leading to instability.

Indeed, tragically, an experimental X-15 aircraft flying this control system crashed in 1966 (cf. Staff of the Flight Research Center [1971]), owing partially to the saturation problems occurring in the high-gain scheme. The roll and pitch axes were controlled by the right and left rear ailerons, using differential and identical commands respectively. The two axes were assumed decoupled for the purpose of control design. However, saturation of the actuators in the pitch axis caused the aircraft to lose controllability in the roll axis (since the ailerons were at maximum deflection). Due to the saturation, the instability remained undetected, and created aerodynamic forces too great for the aircraft to withstand.

Parallel Scheme

As in the series scheme, the desired performance of the closed-loop system is specified through a reference model, and the adaptive system attempts to make the plant output match the reference model output asymptotically. An early reference to this scheme is Osburn, Whitaker, & Kezer [1961]. A block diagram is shown in Figure 0.3. The controller

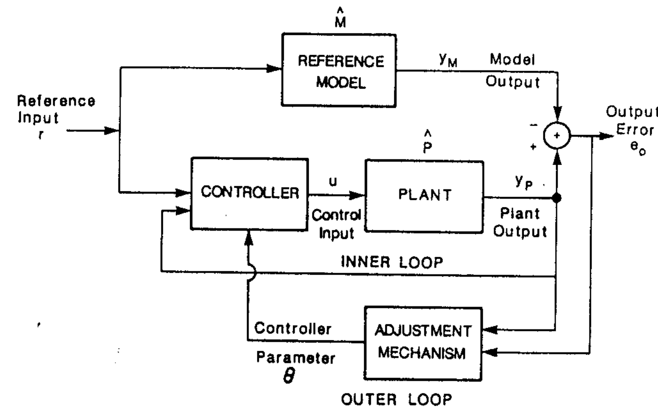

Figure 0.3: Model Reference Adaptive Control—Parallel Scheme

can be thought of as having two loops: an inner or regulator loop that is an ordinary control loop consisting of the plant and regulator, and an outer or adaptation loop that adjusts the parameters of the regulator in such a way as to drive the error between the model output and plant output to zero.

The key problem in the scheme is to obtain an adjustment mechanism that drives the output error $e_0 = y_p - y_m$ to zero. In the earliest applications of this scheme, the following update, called the *gradient update*, was used. Let the vector θ contain the adjustable parameters of the controller. The idea behind the gradient update is to reduce $e_0^2(\theta)$ by adjusting θ along the direction of steepest descent, that is

$$\frac{d\theta}{dt} = -g\frac{\partial}{\partial\theta}(e_0^2(\theta)) \tag{0.2.1}$$

$$= -2g\,e_0(\theta)\frac{\partial}{\partial\theta}(e_0(\theta)) = -2g\,e_0(\theta)\frac{\partial}{\partial\theta}(y_p(\theta)) \tag{0.2.2}$$

where g is a positive constant called the *adaptation gain*.

The interpretation of $e_0(\theta)$ is as follows: it is the output error (also a function of time) obtained by freezing the controller parameter at θ. The gradient of $e_0(\theta)$ with respect to θ is equal to the gradient of y_p with respect to θ, since y_m is independent of θ, and represents the *sensitivity* of the output error to variations in the controller parameter θ.

Several problems were encountered in the usage of the gradient update. The sensitivity function $\partial y_p(\theta)/\partial\theta$ usually depends on the unknown plant parameters, and is consequently unavailable. At this point the so-called *M.I.T. rule*, which replaced the unknown parameters by their estimates at time t, was proposed. Unfortunately, for schemes based on the M.I.T. rule, it is not possible in general to prove closed-loop stability, or convergence of the output error to zero. Empirically, it was observed that the M.I.T. rule performed well when the adaptation gain g and the magnitude of the reference input were small (a conclusion later confirmed analytically by Mareels *et al* [1986]). However, examples of instability could be obtained otherwise (cf. James [1971]).

Parks [1966] found a way of redesigning adaptive systems using Lyapunov theory, so that stable and provably convergent model reference schemes were obtained. The update laws were similar to (0.2.2), with the sensitivity $\partial y_p(\theta)/\partial\theta$ replaced by other functions. The stability and convergence properties of model reference adaptive systems make them particularly attractive and will occupy a lot of our interest in this book.

0.2.3 Self Tuning Regulators

In this technique of adaptive control, one starts from a control design method for known plants. This design method is summarized by a controller structure, and a relationship between plant parameters and controller parameters. Since the plant parameters are in fact unknown, they are obtained using a recursive parameter identification algorithm. The controller parameters are then obtained from the estimates of the plant parameters, in the same way *as if these were the true parameters.* This is usually called a *certainty equivalence principle.*

The resulting scheme is represented on Figure 0.4. An explicit separation between identification and control is assumed, in contrast to the model reference schemes above, where the parameters of the controller are updated directly to achieve the goal of model following. The self tuning approach was originally proposed by Kalman [1958] and clarified by Astrom & Wittenmark [1973]. The controller is called *self tuning*, since it has the ability to tune its own parameters. Again, it can be thought of as having two loops: an inner loop consisting of a conventional controller, but with varying parameters, and an outer loop consisting of an identifier and a design box (representing an on-line solution to a design problem for a system with known parameters) which adjust these controller parameters.

The self tuning regulator is very flexible with respect to its choice of controller design methodology (linear quadratic, minimum variance, gain-phase margin design, ...), and to the choice of identification scheme

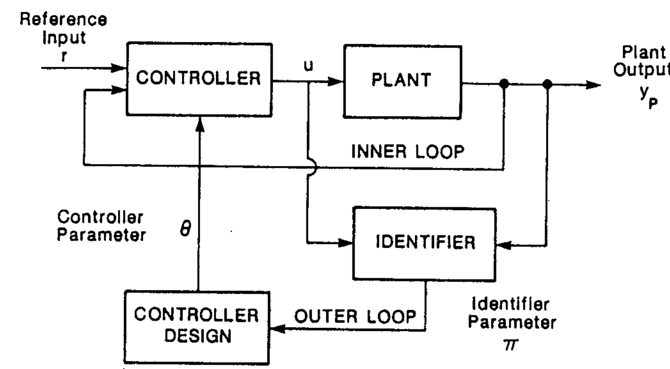

Figure 0.4: Self-tuning Controller

(least squares, maximum likelihood, extended Kalman filtering, ...). The analysis of self tuning adaptive systems is however more complex than the analysis of model reference schemes, due primarily to the (usually nonlinear) transformation from identifier parameters to controller parameters.

Direct and Indirect Adaptive Control

While model reference adaptive controllers and self tuning regulators were introduced as different approaches, the only real difference between them is that model reference schemes are *direct* adaptive control schemes, whereas self tuning regulators are *indirect.* The self tuning regulator first identifies the plant parameters recursively, and then uses these estimates to update the controller parameters through some fixed transformation. The model reference adaptive schemes update the controller parameters directly (no explicit estimate or identification of the plant parameters is made). It is easy to see that the inner or control loop of a self tuning regulator could be the same as the inner loop of a model reference design. Or, in other words, the model reference adaptive schemes can be seen as a special case of the self tuning regulators, with an identity transformation between updated parameters and controller parameters. Through this book, we will distinguish between direct and indirect schemes rather than between model reference and self tuning algorithms.

0.2.4 Stochastic Control Approach

Adaptive controller structures based on model reference or self tuning approaches are based on heuristic arguments. Yet, it would be appealing to obtain such structures from a unified theoretical framework. This can be done (in principle, at least) using stochastic control. The system and its environment are described by a stochastic model, and a criterion is formulated to minimize the expected value of a loss function, which is a scalar function of states and controls. It is usually very difficult to solve stochastic optimal control problems (a notable exception is the linear quadratic gaussian problem). When indeed they can be solved, the optimal controllers have the structure shown in Figure 0.5: an identifier (estimator) followed by a nonlinear feedback regulator.

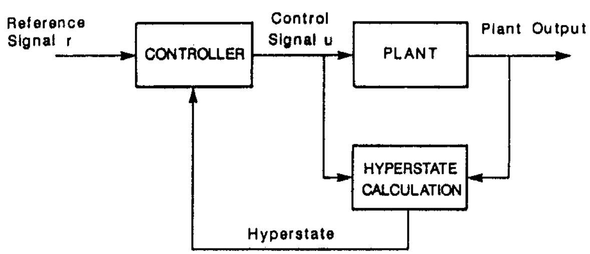

Figure 0.5: "Generic" Stochastic Controller

The estimator generates the conditional probability distribution of the state from the measurements: this distribution is called the *hyperstate* (usually belonging to an infinite dimensional vector space). The self tuner may be thought of as an approximation of this controller, with the hyperstate approximated by the process state and process parameters estimate.

From some limited experience with stochastic control, the following interesting observations can be made of the optimal control law: in addition to driving the plant output to its desired value, the controller introduces *probing* signals which improve the identification and, therefore future control. This, however, represents some cost in terms of control activity. The optimal regulator maintains a balance between the control activity for *learning* about the plant it is controlling and the activity for *controlling* the plant output to its desired value. This property is referred to as *dual control.* While we will not explicitly study stochastic control in this book, the foregoing trade-off will be seen repeatedly: good adaptive control requires correct identification, and for the identification to be complete, the controller signal has to be sufficiently rich to allow for the excitation of the plant dynamics. The

presence of this rich enough excitation may result in poor transient performance of the scheme, displaying the trade-off between learning and control performance.

0.3 A SIMPLE EXAMPLE

Adaptive control systems are difficult to analyze because they are nonlinear, time varying systems, even if the plant that they are controlling is linear, time invariant. This leads to interesting and delicate technical problems. In this section, we will introduce some of these problems with a simple example. We also discuss some of the adaptive schemes of the previous section in this context.

We consider a first order, time invariant, linear system with transfer function

$$\hat{P}(s) = \frac{k_p}{s+a} \tag{0.3.1}$$

where $a > 0$ is known. The gain k_p of the plant is unknown, but its sign is known (say $k_p > 0$). The control objective is to get the plant output to match a model output, where the reference model transfer function is

$$\hat{M}(s) = \frac{1}{s+a} \tag{0.3.2}$$

Only gain compensation—or feedforward control—is necessary, namely a gain θ at the plant input, as is shown on Figure 0.6.

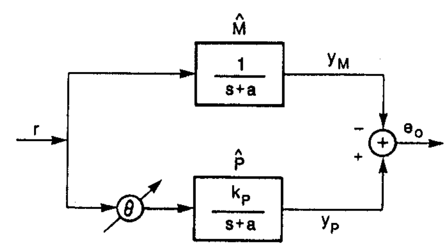

Figure 0.6: Simple Feedforward Controller

Note that, if k_p were known, θ would logically be chosen to be $1/k_p$. We will call

$$\theta^* = \frac{1}{k_p} \tag{0.3.3}$$

the *nominal* value of the parameter θ, that is the value which realizes the output matching objective for all inputs. The design of the various adaptive schemes proceeds as follows.

Gain Scheduling

Let $v(t) \in \mathbb{R}$ be some auxiliary measurement that correlates in known fashion with k_p, say $k_p(t) = f(v(t))$. Then, the gain scheduler chooses at time t

$$\theta(t) = \frac{1}{f(v(t))} \qquad (0.3.4)$$

Model Reference Adaptive Control Using the M.I.T. Rule

To apply the M.I.T. rule, we need to obtain $\partial e_0(\theta)/\partial\theta = \partial y_p(\theta)/\partial\theta$, with the understanding that θ is frozen. From Figure 0.6, it is easy to see that

$$\frac{\partial y_p(\theta)}{\partial\theta} = \frac{k_p}{s+a}(r) = k_p y_m \qquad (0.3.5)$$

We see immediately that the sensitivity function in (0.3.5) depends on the parameter k_p which is unknown, so that $\partial y_p/\partial\theta$ is not available. However, the sign of k_p is known ($k_p > 0$), so that we may merge the constant k_p with the adaptation gain. The M.I.T rule becomes

$$\dot\theta = -g\, e_0 y_m \qquad g > 0 \qquad (0.3.6)$$

Note that (0.3.6) prescribes an update of the parameter θ in the direction opposite to the "correlation" product of e_0 and the model output y_m.

Model Reference Adaptive Control Using the Lyapunov Redesign

The control scheme is exactly as before, but the parameter update law is chosen to make a Lyapunov function decrease along the trajectories of the adaptive system (see Chapter 1 for an introduction to Lyapunov analysis). The plant and reference model are described by

$$\dot y_p = -a\, y_p + k_p \theta\, r \qquad (0.3.7)$$

$$\dot y_m = -a\, y_m + r = -a\, y_m + k_p \theta^* r \qquad (0.3.8)$$

Subtracting (0.3.8) from (0.3.7), we get, with $e_0 = y_p - y_m$

$$\dot e_0 = -a\, e_0 + k_p(\theta - \theta^*)\, r \qquad (0.3.9)$$

Since we would like θ to converge to the nominal value $\theta^* = 1/k_p$, we define the *parameter error* as

$$\phi = \theta - \theta^* \qquad (0.3.10)$$

Note that since θ^* is fixed (though unknown), $\dot\phi = \dot\theta$.

The Lyapunov redesign approach consists in finding an update law so that the Lyapunov function

$$v(e_0, \phi) = e_0^2 + k_p \phi^2 \qquad (0.3.11)$$

is decreasing along trajectories of the error system

$$\begin{aligned} \dot e_0 &= -a\, e_0 + k_p \phi\, r \\ \dot\phi &= \text{update law to be defined} \end{aligned} \qquad (0.3.12)$$

Note that since $k_p > 0$, the function $v(e_0, \phi)$ is a positive definite function. The derivative of v along the trajectories of the error system (0.3.12) is given by

$$\dot v(e_0,\phi)\Big|_{(0.3.12)} = -2a\, e_0^2 + 2k_p e_0 \phi\, r + 2k_p \phi\dot\phi \qquad (0.3.13)$$

Choosing the update law

$$\dot\theta = \dot\phi = -e_0 r \qquad (0.3.14)$$

yields

$$\dot v(e_0, \phi) = -2a\, e_0^2 \le 0 \qquad (0.3.15)$$

thereby guaranteeing that $e_0^2 + k_p\phi^2$ is decreasing along the trajectories of (0.3.12), (0.3.14) and that e_0, and ϕ are bounded. Note that (0.3.14) is similar in form to (0.3.6), with the difference that e_0 is correlated with r rather that y_m. An adaptation gain g may also be included in (0.3.14).

Since $v(e_0, \phi)$ is decreasing and bounded below, it would appear that $e_0 \to 0$ as $t \to \infty$. This actually follows from further analysis, provided that r is bounded (cf. Barbalat's lemma 1.2.1).

Having concluded that $e_0 \to 0$ as $t \to \infty$, what can we say about θ? Does it indeed converge to $\theta^* = 1/k_p$? The answer is that one can not conclude anything about the convergence of θ to θ^* without extra conditions on the reference input. Indeed, if the reference input was a constant zero signal, there would be no reason to expect θ to converge to θ^*. Conditions for parameter convergence are important in adaptive control and will be studied in great detail. An answer to this question for the simple example will be given for the following indirect adaptive control scheme.

Indirect Adaptive Control (Self Tuning)

To be able to effectively compare the indirect scheme with the direct schemes given before, we will assume that the control objective is still model matching, with the same model as above. Figure 0.7 shows an indirect or self tuning type of model reference adaptive controller.

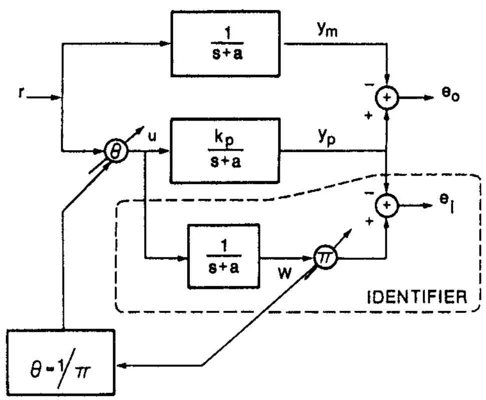

Figure 0.7: A Simple Indirect Controller

The identifier contains an identifier parameter $\pi(t)$ that is an estimate of the unknown plant parameter k_p. Therefore, we define $\pi^* = k_p$. The controller parameter is chosen following the certainty equivalence principle: since $\theta^* = 1/k_p$ and $\pi^* = k_p$, we let $\theta(t) = 1/\pi(t)$. The hope is that, as $t \to \infty$, $\pi(t) \to k_p$, so that $\theta(t) \to 1/k_p$.

The update law now is an update law for the identifier parameter $\pi(t)$. There are several possibilities at this point, and we proceed to derive one of them. Define the identifier parameter error

$$\psi(t) \;:=\; \pi(t) - \pi^* \tag{0.3.16}$$

and let

$$w \;=\; \frac{1}{s+a}\,(\theta r) \;=\; \frac{1}{s+a}\,(u) \tag{0.3.17}$$

The signal w may be obtained by stable filtering of the input u, since $a > 0$ is known. The update law is based on the identifier error

$$e_i \;:=\; \pi w - y_p \tag{0.3.18}$$

Equation (0.3.18) is used in the actual implementation of the algorithm. For the analysis, note that

$$y_p \;=\; \frac{k_p}{s+a}\,(\theta r) \;=\; \pi^* w \tag{0.3.19}$$

so that

$$e_i \;=\; \psi w \tag{0.3.20}$$

Consider the update law

$$\dot{\pi} \;=\; \dot{\psi} \;=\; -g\, e_i\, w \qquad g > 0 \tag{0.3.21}$$

and let the Lyapunov function

$$v \;=\; \psi^2 \tag{0.3.22}$$

This Lyapunov function has the special form of the norm square of the identifier parameter error. Its derivative along the trajectories of the adaptive system is

$$\dot{v} \;=\; -g\,\psi^2 w^2 \tag{0.3.23}$$

Therefore, the update law causes a decreasing parameter error and all signals remain bounded.

The question of parameter convergence can be answered quite simply in this case. Note that (0.3.20), (0.3.21) represent the first order linear time varying system

$$\dot{\psi} \;=\; -g\, w^2 \psi \tag{0.3.24}$$

which may be explicitly integrated to get

$$\psi(t) \;=\; \psi(0)\, \exp\left(-g \int_0^t w^2(\tau)\, d\tau\right) \tag{0.3.25}$$

It is now easy to see that if

$$\int_0^t w^2(\tau)\, d\tau \;\to\; \infty \qquad \text{as } t \to \infty \tag{0.3.26}$$

then $\psi(t) \to 0$, so that $\pi(t) \to \pi^*$ and $\theta(t) \to 1/k_p$, yielding the desired controller. The condition (0.3.26) is referred to as an *identifiability* condition and is much related to the so-called *persistency of excitation* that will be discussed in Chapter 2. It is easily seen that, in particular, it excludes signals which tend to zero as $t \to \infty$.

The difficulty with (0.3.26) is that it depends on w, which in turn depends on u and therefore on both θ and r. Converting it into a condition on the exogenous reference input $r(t)$ only is another of the problems which we will discuss in the following chapters.

The foregoing simple example showed that even when simple feedforward control of a linear, time invariant, first order plant was involved, the analysis of the resulting closed-loop dynamics could be involved: the equations were *time-varying, linear* equations. Once feedback control is involved, the equations become nonlinear and time varying.

CHAPTER 1
PRELIMINARIES

This chapter introduces the notation used in this book, as well as some basic definitions and results. The material is provided mostly for reference. It may be skipped in a first reading, or by the reader familiar with the results.

The notation used in the adaptive systems literature varies widely. We elected to use a notation close to that of Narendra & Valavani [1978], and Narendra, Lin, & Valavani [1980], since many connections exist between this work and their results. We will refer to texts such as Desoer & Vidyasagar [1975], and Vidyasagar [1978] for standard results, and this chapter will concentrate on the definitions used most often, and on nonstandard results.

1.1 NOTATION

Lower case letters are used to denote scalars or vectors. Upper case letters are used to denote matrices, operators, or sets. When $u(t)$ is a function of time, $\hat{u}(s)$ denotes its Laplace transform. Without ambiguity, we will drop the arguments, and simply write u and $\hat{u}$. Rational transfer functions of linear time invariant (LTI) systems will be denoted using upper case letters, for example, $\hat{H}(s)$ or $\hat{H}$. Polynomials in s will be denoted using lower case letters, for example, $\hat{n}(s)$ or simply $\hat{n}$. Thus, we may have $\hat{H} = \hat{n}/\hat{d}$, where $\hat{H}$ is both the ratios of polynomials in s and an operator in the Laplace transform domain. Sometimes, the time domain and the Laplace transform domain will be mixed, and parentheses will determine the sense to be made of an expression. For example, $\hat{H}(u)$ or $\hat{H}\,\hat{u}$ is the output of the LTI system $\hat{H}$ with input u. $\hat{H}(u)v$ is $\hat{H}(u)$ multiplied by v in the time domain, while $\hat{H}(uv)$ is $\hat{H}$ operating on the product $u(t)\,v(t)$.

1.2 L_p SPACES, NORMS

We denote by $|x|$ the absolute value of x if x is a scalar and the euclidean norm of x if x is a vector. The notation $\| \ \|$ will be used to denote the induced norm of an operator, in particular the induced matrix norm

$$\|A\| = \sup_{|x|=1} |Ax| \tag{1.2.1}$$

and for functions of time, the notation is used for the L_p norm

$$\|u\|_p = \left(\int_0^\infty |u(\tau)|^p \, d\tau \right)^{1/p} \tag{1.2.2}$$

for $p \in [1, \infty)$, while

$$\|u\|_\infty = \sup_{t \geq 0} |u(t)| \tag{1.2.3}$$

and we say that $u \in L_p$ when $\|u\|_p$ exists. When p is omitted, $\|u\|$ denotes the L_2 norm. Truncated functions are defined as

$$f_s(t) = f(t) \qquad t \leq s$$
$$\qquad = 0 \qquad t > s \tag{1.2.4}$$

and the extended L_p spaces are defined by

$$L_{pe} = \{ f \mid \text{ for all } s < \infty, f_s \in L_p \} \tag{1.2.5}$$

For example, e^t does not belong to L_∞, but $e^t \in L_{\infty e}$. When $u \in L_{\infty e}$, we have

$$\|u_t\|_\infty = \sup_{\tau \leq t} |u(\tau)| \tag{1.2.6}$$

A function f may belong to L_1 and not be bounded. Conversely, a bounded function need not belong to L_1. However, if $f \in L_1 \cap L_\infty$, then $f \in L_p$ for all $p \in [1, \infty]$ (cf. Desoer & Vidyasagar [1975], p. 17).

Also, $f \in L_p$ does not imply that $f \to 0$ as $t \to \infty$. This is not even guaranteed if f is bounded. However, note the following results.

Lemma 1.2.1 Barbalat's Lemma

If $f(t)$ is a uniformly continuous function, such that $\lim_{t \to \infty} \int_0^t f(\tau)\, d\tau$ exists and is finite,

Then $f(t) \to 0$ as $t \to \infty$.

Proof of Lemma 1.2.1 cf. Popov [1973] p. 211.

Corollary 1.2.2

If $g, \dot{g} \in L_\infty$, and $g \in L_p$, for some $p \in [1, \infty)$,

Then $g(t) \to 0$ as $t \to \infty$.

Proof of Corollary 1.2.2 lower case

Direct from lemma 1.2.1, with $f = |g|^p$, since $g, \dot{g}$ bounded implies that f is uniformly continuous. □

1.3 POSITIVE DEFINITE MATRICES

Positive definite matrices are frequently found in work on adaptive systems. We summarize here several facts that will be useful. We consider real matrices. Recall that a scalar u, or a function of time $u(t)$, is said to be *positive* if $u \geq 0$, or $u(t) \geq 0$ for all t. It is *strictly positive* if $u > 0$, or, for some $\alpha > 0$, $u(t) \geq \alpha$ for all t. A square matrix $A \in R^{n \times n}$ is *positive semidefinite* if $x^T A x \geq 0$ for all x. It is *positive definite* if, for some $\alpha > 0$, $x^T A x \geq \alpha x^T x = \alpha |x|^2$ for all x. Equivalently, we can require $x^T A x \geq \alpha$ for all x such that $|x| = 1$. The matrix A is negative semidefinite if $-A$ is positive semidefinite and for symmetric matrices, we write $A \geq B$ if $A - B \geq 0$. Note that a matrix can be neither positive semidefinite nor negative semidefinite, so that this only establishes a partial order on symmetric matrices.

The eigenvalues of a positive semidefinite matrix lie in the closed right-half plane (RHP), while those of a positive definite matrix lie in the open RHP. If $A \geq 0$ and $A = A^T$, then A is *symmetric positive semidefinite.* In particular, if $A \geq 0$, then $A + A^T$ is symmetric positive semidefinite. The eigenvalues of a symmetric matrix are all real. Such a matrix also has n orthogonal eigenvectors, so that we can decompose A as

$$A = U^T \Lambda U \tag{1.3.1}$$

where U is the matrix of eigenvectors satisfying $U^T U = I$ (that is, U is a *unitary* matrix), and Λ is a diagonal matrix composed of the eigenvalues of A. When $A \geq 0$, the square root matrix $\Lambda^{1/2}$ is a diagonal matrix composed of the square roots of the eigenvalues of A, and

$$A^{1/2} = U^T \Lambda^{1/2} U \tag{1.3.2}$$

is the square root matrix of A, with $A = A^{1/2} \cdot A^{1/2}$ and $(A^{1/2})^T = A^{1/2}$.

If $A \geq 0$ and $B \geq 0$, then $A + B \geq 0$ but it is not true in general that $A \cdot B \geq 0$. However, if A, B are symmetric, positive semidefinite matrices, then AB—although not necessarily symmetric, or positive semidefinite—has all eigenvalues real positive.

Another property of symmetric, positive semidefinite matrices, following from (1.3.1), is

$$\lambda_{\min}(A) | x |^2 \leq x^T A x \leq \lambda_{\max}(A) | x |^2 \tag{1.3.3}$$

This simply follows from the fact that $x^T A x = x^T U^T \Lambda U x = z^T \Lambda z$ and $| z |^2 = z^T z = | x |^2$. We also have that

$$\| A \| = \lambda_{\max}(A) \tag{1.3.4}$$

and, when A is positive definite

$$\| A^{-1} \| = 1/\lambda_{\min}(A) \tag{1.3.5}$$

1.4 STABILITY OF DYNAMIC SYSTEMS

1.4.1 Differential Equations

This section is concerned with differential equations of the form

$$\dot{x} = f(t, x) \qquad x(t_0) = x_0 \tag{1.4.1}$$

where $x \in \mathbb{R}^n, t \geq 0$.

The system defined by (1.4.1) is said to be *autonomous*, or *time-invariant*, if f does not depend on t, and *non autonomous*, or *time-varying*, otherwise. It is said to be *linear* if $f(t,x) = A(t)x$ for some $A(.) : \mathbb{R}_+ \to \mathbb{R}^{n \times n}$ and *nonlinear* otherwise.

We will always assume that $f(t,x)$ is *piecewise continuous* with respect to t. By this, we mean that there are only a finite number of discontinuity points in any compact set.

We define by B_h the closed ball of radius h centered at 0 in $\mathbb{R}^n$. Properties will be said to be true:

- *locally*, if true for all x_0 in some ball B_h.

- *globally*, if true for all $x_0 \in \mathbb{R}^n$.
- *in any closed ball*, if true for all $x_0 \in B_h$, with h arbitrary.
- *uniformly*, if true for all $t_0 \geq 0$.

By default, properties will be true locally.

Lipschitz Condition and Consequences

The function f is said to be *Lipschitz* in x if, for some $h > 0$, there exists $l \geq 0$ such that

$$| f(t, x_1) - f(t, x_2) | \leq l | x_1 - x_2 | \tag{1.4.2}$$

for all $x_1, x_2 \in B_h, t \geq 0$. The constant l is called the *Lipschitz constant.* This defines locally Lipschitz functions. Globally Lipschitz functions satisfy (1.4.2) for all $x_1, x_2 \in \mathbb{R}^n$, while functions that are Lipschitz in any closed ball satisfy (1.4.2) for all $x_1, x_2 \in B_h$, with l possibly depending on h. The Lipschitz property is by default assumed to be satisfied uniformly, that is, l does not depend on t.

If f is Lipschitz in x, then it is continuous in x. On the other hand, if f has continuous and bounded partial derivatives in x, then it is Lipschitz. More formally, we denote

$$D_2 f := \left[\frac{\partial f_i}{\partial x_j} \right] \tag{1.4.3}$$

so that if $\| D_2 f \| \leq l$, then f is Lipschitz with constant l.

From the theory of ordinary differential equations (cf. Coddington & Levinson [1955]), it is known that f locally bounded, and f locally Lipschitz in x imply the existence and uniqueness of the solutions of (1.4.1) on some time interval (for as long as $x \in B_h$).

Definition Equilibrium Point

x is called an *equilibrium point* of (1.4.1), if $f(t, x) = 0$ for all $t \geq 0$.

By translating the origin to an equilibrium point x_0, we can make the origin 0 an equilibrium point. This is of great notational help, and we will assume henceforth that 0 is an equilibrium point of (1.4.1).

Proposition 1.4.1

If $x = 0$ is an equilibrium point of (1.4.1), f is Lipschitz in x with constant l and is piecewise continuous with respect to t

Then the solution $x(t)$ of (1.4.1) satisfies

$$|x_0|\, e^{l(t-t_0)} \;\geq\; |x(t)| \;\geq\; |x_0|\, e^{-l(t-t_0)} \tag{1.4.4}$$

as long as $x(t)$ remains in B_h.

Proof of Proposition 1.4.1

Note that $|x|^2 = x^T x$ implies that

$$\left|\frac{d}{dt}|x|^2\right| = 2|x|\left|\frac{d}{dt}|x|\right|$$

$$= 2\left|x^T \frac{d}{dt}x\right| \leq 2|x|\left|\frac{d}{dt}x\right| \tag{1.4.5}$$

so that

$$\left|\frac{d}{dt}|x|\right| \leq \left|\frac{d}{dt}x\right| \tag{1.4.6}$$

Since f is Lipschitz

$$-l|x| \leq \frac{d}{dt}|x| \leq l|x| \tag{1.4.7}$$

and there exists a positive function $s(t)$ such that

$$\frac{d}{dt}|x| = -l|x| + s \tag{1.4.8}$$

Solving (1.4.8)

$$|x(t)| = |x_0|\, e^{-l(t-t_0)} + \int_0^t e^{-l(t-\tau)} s(\tau)\, d\tau$$

$$\geq |x_0|\, e^{-l(t-t_0)} \tag{1.4.9}$$

The other inequality follows similarly from (1.4.7). □

Proposition 1.4.1 implies that solutions starting inside B_h will remain inside B_h for at least a finite time interval. Or, conversely, given a time interval, the solutions will remain in B_h provided that the initial conditions are sufficiently small. Also, f globally Lipschitz implies that $x \in L_{\infty e}$. Proposition 1.4.1 also says that x cannot tend to zero faster than exponentially.

The following lemma is an important result generalizing the well-known Bellman-Gronwall lemma (Bellman [1943]). The proof is similar to the proof of proposition 1.4.1, and is left to the appendix.

Lemma 1.4.2 Bellman-Gronwall Lemma

Let $x(.), a(.), u(.) : \mathbb{R}_+ \to \mathbb{R}_+$. Let $T \geq 0$.

If

$$x(t) \leq \int_0^t a(\tau)x(\tau)\, d\tau + u(t) \tag{1.4.10}$$

for all $t \in [0, T]$

Then

$$x(t) \leq \int_0^t a(\tau)u(\tau)\, e^{\int_\tau^t a(\sigma)d\sigma}\, d\tau + u(t) \tag{1.4.11}$$

for all $t \in [0, T]$

When $u(.)$ is differentiable

$$x(t) \leq u(0)\, e^{\int_0^t a(\sigma)d\sigma} + \int_0^t \dot{u}(\tau)\, e^{\int_\tau^t a(\sigma)d\sigma}\, d\tau \tag{1.4.12}$$

for all $t \in [0, T]$

Proof of Lemma 1.4.2 in Appendix.

1.4.2 Stability Definitions

Informally, $x = 0$ is a *stable* equilibrium point, if the trajectory $x(t)$ remains close to 0 if the initial condition x_0 is close to 0. More precisely, we say

Definition Stability in the Sense of Lyapunov

$x = 0$ is called a *stable* equilibrium point of (1.4.1), if, for all $t_0 \geq 0$ and $\epsilon > 0$, there exists $\delta(t_0, \epsilon)$ such that

$$|x_0| < \delta(t_0, \epsilon) \implies |x(t)| < \epsilon \qquad \text{for all } t \geq t_0$$

where $x(t)$ is the solution of (1.4.1) starting from x_0 at t_0.

Definition Uniform Stability

$x = 0$ is called a *uniformly stable* equilibrium point of (1.4.1) if, in the preceding definition, δ can be chosen independent of t_0.

Intuitively, this definition captures the notion that the equilibrium point is not getting progressively less stable with time. Stability is a very mild requirement for an equilibrium point. In particular, it does not require that trajectories starting close to the origin tend to the origin asymptotically. That property is made precise in the following definition.

Definition Asymptotic Stability

$x = 0$ is called an *asymptotically stable* equilibrium point of (1.4.1), if

(a) $x = 0$ is a stable equilibrium point of (1.4.1),

(b) $x = 0$ is *attractive*, that is, for all $t_0 \geq 0$, there exists $\delta(t_0)$, such that

$$|x_0| < \delta \implies \lim_{t \to \infty} |x(t)| = 0$$

Definition Uniform Asymptotic Stability (u.a.s.)

$x = 0$ is called a *uniformly asymptotically stable (u.a.s.)* equilibrium point of (1.4.1), if

(a) $x = 0$ is a uniformly stable equilibrium point of (1.4.1),

(b) the trajectory $x(t)$ converges to 0 uniformly in t_0. More precisely, there exists $\delta > 0$ and a function $\gamma(\tau, x_0)$: $\mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$, such that $\lim_{\tau \to \infty} \gamma(\tau, x_0) = 0$ for all x_0 and

$$|x_0| < \delta \implies |x(t)| \leq \gamma(t - t_0, x_0) \qquad \text{for all } t \geq 0$$

The previous definitions are *local*, since they concern neighborhoods of the equilibrium point. *Global* asymptotic stability is defined as follows.

Definition Global Asymptotic Stability

$x = 0$ is called a *globally asymptotically stable* equilibrium point of (1.4.1), if it is asymptotically stable and $\lim_{t \to \infty} |x(t)| = 0$, for all $x_0 \in \mathbb{R}^n$.

Global u.a.s. is defined likewise. Note that the speed of convergence is not quantified in the definitions of asymptotic stability. In the following definition, the convergence to zero is required to be at least exponential.

Definition Exponential Stability, Rate of Convergence

$x = 0$ is called an *exponentially stable* equilibrium point of (1.4.1) if there exist m, $\alpha > 0$ such that the solution $x(t)$ satisfies

$$|x(t)| \leq m e^{-\alpha(t - t_0)} |x_0| \qquad (1.4.13)$$

for all $x_0 \in B_h$, $t \geq t_0 \geq 0$. The constant α is called the *rate of convergence*.

Global exponential stability means that (1.4.13) is satisfied for any $x_0 \in \mathbb{R}^n$. Exponential stability in any closed ball is similar except that m and α may be functions of h. Exponential stability is assumed to be uniform with respect to t_0. It will be shown that uniform asymptotic stability is equivalent to exponential stability for linear systems (see Section 1.5.2), but it is not true in general.

1.4.3 Lyapunov Stability Theory

We now review some of the key concepts and results of Lyapunov stability theory for ordinary differential equations of the form (1.4.1). A more complete development is available, for instance, in the texts by Hahn [1967] and Vidyasagar [1978].

The so-called Lyapunov second method enables one to determine the nature of stability of an equilibrium point of (1.4.1) without explicitly integrating the differential equation. The method is basically a generalization of the idea that if some "measure of the energy" associated with a system is decreasing, then the system will tend to its equilibrium. To make this notion precise, we need to define exactly what we mean by a "measure of energy," that is, energy functions. For this, we first define class K functions (Hahn [1967], p. 7).

Definition Class K Functions

A function $\alpha(\epsilon) : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to class K (denoted $\alpha(.) \in K$), if it is continuous, strictly increasing, and $\alpha(0) = 0$.

Definition Locally Positive Definite Functions

A continuous function $v(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is called a *locally positive definite function (l.p.d.f.)* if, for some $h > 0$, and some $\alpha(.) \in K$

$$v(t, 0) = 0 \quad \text{and} \quad v(t, x) \geq \alpha(|x|) \qquad \text{for all } x \in B_h, t \geq 0$$

An l.p.d.f. is locally like an "energy function." Functions which are globally like "energy functions" are called positive definite functions (p.d.f.) and are defined as follows.

Definition Positive Definite Functions

A continuous function $v(t, x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ is called a *positive definite function (p.d.f.)*, if for some $\alpha(.) \in K$

$$v(t, 0) = 0 \quad \text{and} \quad v(t, x) \geq \alpha(|x|) \qquad \text{for all } x \in \mathbb{R}^n, t \geq 0$$

and the function $\alpha(p) \to \infty$ as $p \to \infty$.

In the definitions of l.p.d.f. and p.d.f. functions, the energy like functions are not bounded from above as t varies. This follows in the next definition.

Definition Decrescent Function

The function $v(t,x)$ is called *decrescent*, if there exists a function $\beta(.) \in K$, such that

$$v(t,x) \le \beta(\|x\|) \qquad \text{for all } x \in B_h, t \ge 0$$

Examples

Following are several examples of functions, and their membership in the various classes:

$v(t,x) = |x|^2$: p.d.f., decrescent

$v(t,x) = x^T P x$, with $P > 0$: p.d.f., decrescent

$v(t,x) = (t+1)|x|^2$: p.d.f.

$v(t,x) = e^{-t}|x|^2$: decrescent

$v(t,x) = \sin^2(|x|^2)$: l.p.d.f., decrescent

Lyapunov Stability Theorems

Generally speaking, the theorems state that when $v(t,x)$ is a p.d.f., or an l.p.d.f., and $dv/dt(t,x) \le 0$, then we can conclude the stability of the equilibrium point. The derivative of v is taken along the trajectories of (1.4.1); that is,

$$\left.\frac{dv(t,x)}{dt}\right|_{(1.4.1)} = \frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} f(t,x) \qquad (1.4.14)$$

Theorem 1.4.3 Basic Theorems of Lyapunov

Let $v(t,x)$ be continuously differentiable:

Then

Conditions on v(t, x)	Conditions on − v̇(t, x)	Conclusions
l.p.d.f.	≥ 0 locally	stable
l.p.d.f., decrescent	≥ 0 locally	uniformly stable
l.p.d.f.	l.p.d.f.	asymptotically stable
l.p.d.f., decrescent	l.p.d.f.	uniformly asymptotically stable
p.d.f., decrescent	p.d.f.	globally u.a.s

Proof of Theorem 1.4.3 cf. Vidyasagar [1978], p. 148 and after.

Example

We refer to Vidyasagar [1978] for examples of application of the theorems. An interesting example is the second order system

$$\dot{x}_1 = x_1(x_1^2 + x_2^2 - 1) - x_2$$

$$\dot{x}_2 = x_1 + x_2(x_1^2 + x_2^2 - 1) \qquad (1.4.15)$$

with the p.d.f.

$$v(x_1, x_2) = x_1^2 + x_2^2 \qquad (1.4.16)$$

The derivative of v is given by

$$\dot{v}(x_1, x_2) = 2(x_1^2 + x_2^2)(x_1^2 + x_2^2 - 1) \qquad (1.4.17)$$

so that $-\dot{v}$ is an l.p.d.f. (in B_h, where $h < 1$), establishing the local asymptotic stability of the origin. The origin is, in fact, not globally asymptotically stable: the system can be shown to have a circular limit cycle of radius 1 (by changing to polar coordinates).

From (1.4.17), we can also show that for all $x \in B_h$

$$\dot{v} \le -2(1-h^2)v \qquad (1.4.18)$$

so that

$$v(t) \le v(0)e^{-2(1-h^2)t}$$

$$|x(t)| \le |x(0)|\, e^{-(1-h^2)t} \quad \text{for all } t \ge 0 \qquad (1.4.19)$$

and, in fact, $x = 0$ is a locally exponentially stable equilibrium point. Note that the Lyapunov function is in fact the squared norm of the state, a situation that will often occur in the sequel.

Comments

The theorems of Lyapunov give sufficient conditions guaranteeing the stability of the system (1.4.1). It is a remarkable fact that the *converse* of theorem 1.4.3 is also true: for example, if an equilibrium point is stable, there exists an l.p.d.f. $v(t,x)$ with $\dot{v}(t,x) \le 0$. The usefulness of theorem 1.4.3 and its converse is limited by the fact that there is no general (and computationally non-intensive) prescription for generating the Lyapunov functions. A significant exception to this concerns exponentially stable systems, which are the topic of the following section.

1.5 EXPONENTIAL STABILITY THEOREMS

We will pay special attention to exponential stability for two reasons. When considering the convergence of adaptive algorithms, exponential stability means convergence, and the rate of convergence is a useful measure of how fast estimates converge to their nominal values. In Chapter 5, we will also observe that exponentially stable systems possess at least some tolerance to perturbations, and are therefore desirable in engineering applications.

1.5.1 Exponential Stability of Nonlinear Systems

The following theorem will be useful in proving several results and relates exponential stability to the existence of a specific Lyapunov function.

Theorem 1.5.1 Converse Theorem of Lyapunov

Assume that $f(t,x) : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ has continuous and bounded first partial derivatives in x and is piecewise continuous in t for all $x \in B_h$, $t \ge 0$. Then, the following statements are equivalent:

(a) $x = 0$ is an *exponentially stable* equilibrium point of

$$\dot{x} = f(t,x) \qquad x(t_0) = x_0 \tag{1.5.1}$$

(b) There exists a function $v(t,x)$, and some strictly positive constants h', $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, such that, for all $x \in B_{h'}$, $t \ge 0$

$$\alpha_1 |x|^2 \le v(t,x) \le \alpha_2 |x|^2 \tag{1.5.2}$$

$$\left. \frac{dv(t,x)}{dt} \right|_{(1.5.1)} \le -\alpha_3 |x|^2 \tag{1.5.3}$$

$$\left| \frac{\partial v(t,x)}{\partial x} \right| \le \alpha_4 |x| \tag{1.5.4}$$

Comments

Again, the derivative in (1.5.3) is a derivative taken along the trajectories of (1.5.1), that is

$$\left. \frac{dv(t,x)}{dt} \right|_{(1.5.1)} = \frac{\partial v(t,x)}{\partial t} + \frac{\partial v(t,x)}{\partial x} f(t,x) \tag{1.5.5}$$

This means that we consider x to be a function of t to calculate the derivative along the trajectories of (1.5.1) passing through x at t. It does *not* require of x to be the solution $x(t)$ of (1.5.1) starting at $x(t_0)$.

Theorem 1.5.1 can be found in Krasovskii [1963] p. 60, and Hahn [1967] p. 273. It is known as one of the converse theorems. The proof of the theorem is constructive: it provides an explicit Lyapunov function $v(t,x)$. This is a rather unusual circumstance, and makes the theorem particularly valuable. In the proof, we derive explicit values of the constants involved in (1.5.2)–(1.5.4).

Proof of Theorem 1.5.1

(a) implies (b).

(i) Denote by $p(\tau, x, t)$ the solution at time τ of (1.5.1) starting at $x(t)$, t, and define

$$v(t,x) = \int_t^{t+T} |p(\tau,x,t)|^2 d\tau \tag{1.5.6}$$

where $T > 0$ will be defined in (ii). From the exponential stability and the Lipschitz condition

$$m|x|\, e^{-\alpha(\tau-t)} \ge |p(\tau,x,t)| \ge |x|\, e^{-l(\tau-t)} \tag{1.5.7}$$

and inequality (1.5.2) follows with

$$\alpha_1 := (1 - e^{-2lT})/2l \qquad \alpha_2 := m^2(1 - e^{-2\alpha T})/2\alpha \tag{1.5.8}$$

(ii) Differentiating (1.5.6) with respect to t, we obtain

$$\frac{dv(t,x)}{dt} = |p(t+T,x,t)|^2 - |p(t,x,t)|^2 + \int_t^{t+T} \frac{d}{dt}(|p(\tau,x,t)|^2)\, d\tau \tag{1.5.9}$$

Note that d/dt is a derivative with respect to the *initial* time t and is taken along the trajectories of (1.5.1) By definition of the solution p,

$$p(\tau, x(t+\Delta t), t+\Delta t) = p(\tau, x(t), t) \tag{1.5.10}$$

for all Δt, so that the term in the integral is identically zero over $[t, t+T]$. The second term in the right-hand side of (1.5.9) is simply $|x|^2$, while the first is related to $|x|^2$ by the assumption of exponential stability. It follows that

$$\frac{dv(t,x)}{dt} \leq -(1-m^2e^{-2\alpha T})|x|^2 \tag{1.5.11}$$

Inequality (1.5.3) follows, provided that $T > (1/\alpha)\ln m$ and

$$\alpha_3 := 1 - m^2e^{-2\alpha T} \tag{1.5.12}$$

(iii) Differentiating (1.5.6) with respect to x_i, we have

$$\frac{\partial v(t,x)}{\partial x_i} = 2\int_t^{t+T} \sum_{j=1}^{n} p_j(\tau,x,t)\frac{\partial p_j(\tau,x,t)}{\partial x_i}\, d\tau \tag{1.5.13}$$

Under the assumptions, the partial derivative of the solution with respect to the initial conditions satisfies

$$\begin{aligned} \frac{d}{d\tau}\left[\frac{\partial p_j(\tau,x,t)}{\partial x_i}\right] &= \frac{\partial}{\partial x_i}\left[\frac{d}{d\tau}\, p_j(\tau,x,t)\right] \\ &= \frac{\partial}{\partial x_i}\left[f_j(\tau, p(\tau,x,t))\right] \\ &= \sum_{k=1}^{n} \frac{\partial f_j}{\partial x_k}\bigg|_{\tau, p(\tau,x,t)} \cdot \frac{\partial p_k(\tau,x,t)}{\partial x_i} \end{aligned} \tag{1.5.14}$$

(except possibly at points of discontinuity of $f(\tau,x)$). Denote

$$\begin{aligned} Q_{ij}(\tau,x,t) &:= \partial p_i(\tau,x,t)/\partial x_j \\ A_{ij}(x,t) &:= \partial f_i(t,x)/\partial x_j \end{aligned} \tag{1.5.15}$$

so that (1.5.14) becomes

$$\frac{d}{d\tau}\, Q(\tau,x,t) = A(p(\tau,x,t),\tau)\cdot Q(\tau,x,t) \tag{1.5.16}$$

Equation (1.5.16) defines $Q(\tau,x,t)$, when integrated from $\tau = t$ to $\tau = t+T$, with initial conditions $Q(t,x,t) = I$. Thus, $Q(\tau,x,t)$ is the *state transition matrix* (cf. Section 1.5.2) associated with the time varying matrix $A(p(\tau,x,t),\tau)$. By assumption, $\| A(.,.)\| \leq k$ for some k, so that

$$\| Q(\tau,x,t)\| \leq e^{k(\tau-t)} \tag{1.5.17}$$

and, using the exponential stability again, (1.5.14) becomes

$$\left|\frac{\partial v(t,x)}{\partial x}\right| \leq 2\int_t^{t+T} m|x|\, e^{(k-\alpha)(\tau-t)}d\tau \tag{1.5.18}$$

which is (1.5.4) if we define

$$\alpha_4 := 2m(e^{(k-\alpha)T} - 1)/(k-\alpha) \tag{1.5.19}$$

Note that the function $v(t,x)$ is only defined for $x \in B_{h'}$ with $h' = h/m$, if we wish to guarantee that $p(\tau,x,t) \in B_h$ for all $\tau \geq t$.

(b) implies (a)

This direction is straightforward, using only (1.5.2)–(1.5.3), and we find

$$m := \left[\frac{\alpha_2}{\alpha_1}\right]^{\frac{1}{2}} \qquad \alpha := \frac{1}{2}\frac{\alpha_3}{\alpha_2} \tag{1.5.20}$$

□

Comments

The Lyapunov function $v(t,x)$ can be interpreted as an average of the squared norm of the state along the solutions of (1.5.1). This approach is actually the basis of exact proofs of exponential convergence presented in Sections 2.5 and 2.6 for identification algorithms. On the other hand, the approximate proofs presented in Chapter 4 rely on methods for averaging *the differential system* itself. Then the norm squared of the state itself becomes a Lyapunov function, from which the exponential convergence can be deduced.

Theorem 1.5.1 is mostly useful to establish the existence of the Lyapunov function corresponding to exponentially stable systems. To establish exponential stability from a Lyapunov function, the following theorem will be more appropriate. Again, the derivative is to be taken along the trajectories of (1.5.1).

Theorem 1.5.2 Exponential Stability Theorem

If there exists a function $v(t,x)$, and strictly positive constants α_1, α_2, α_3, and δ, such that for all $x \in B_h, t \geq 0$

$$\alpha_1|x|^2 \leq v(t,x) \leq \alpha_2|x|^2 \tag{1.5.21}$$

$$\frac{d}{dt}\, v(t,x(t))\bigg|_{(1.5.1)} \leq 0 \tag{1.5.22}$$

$$\int_t^{t+\delta} \frac{d}{d\tau} v(\tau, x(\tau)) \bigg|_{(1.5.1)} d\tau \leq -\alpha_3 |x(t)|^2 \qquad (1.5.23)$$

Then $x(t)$ converges exponentially to 0.

Proof of Theorem 1.5.2

From (1.5.23)

$$v(t, x(t)) - v(t+\delta, x(t+\delta)) \geq (\alpha_3/\alpha_2)\, v(t, x(t)) \qquad (1.5.24)$$

for all $t \geq 0$, so that

$$v(t+\delta, x(t+\delta)) \leq (1 - \alpha_3/\alpha_2)\, v(t, x(t)) \qquad (1.5.25)$$

for all $t \geq 0$. From (1.5.22)

$$v(t_1, x(t_1)) \leq v(t, x(t)) \qquad \text{for all } t_1 \in [t, t+\delta] \qquad (1.5.26)$$

Choose for t the sequence t_0, $t_0+\delta$, $t_0+2\delta, \ldots$ so that $v(t, x(t))$ is bounded by a staircase $v(t_0, x(t_0))$, $v(t_0+\delta, x(t_0+\delta)), \ldots$ where the steps are related in geometric progression through (1.5.24). It follows that

$$v(t, x(t)) \leq m_v e^{-\alpha_v (t - t_0)} v(t_0, x(t_0)) \qquad (1.5.27)$$

for all $t \geq t_0 \geq 0$, where

$$m_v = \frac{1}{(1 - \alpha_3/\alpha_2)} \qquad \alpha_v = \frac{1}{\delta} \ln \left[\frac{1}{(1 - \alpha_3/\alpha_2)} \right] \qquad (1.5.28)$$

Similarly,

$$|x(t)| \leq m\, e^{-\alpha(t - t_0)} |x(t_0)| \qquad (1.5.29)$$

where

$$m = \left[\frac{\alpha_2}{\alpha_1} \frac{1}{1 - \alpha_3/\alpha_2} \right]^{\frac{1}{2}} \qquad \alpha = \frac{1}{2\delta} \ln \left[\frac{1}{1 - \alpha_3/\alpha_2} \right] \qquad (1.5.30)$$

□

1.5.2 Exponential Stability of Linear Time-Varying Systems

We now restrict our attention to linear time-varying (LTV) systems of the form

$$\dot{x} = A(t)x \qquad x(t_0) = x_0 \qquad (1.5.31)$$

where $A(t) \in \mathbb{R}^{n \times n}$ is a piecewise continuous function belonging to $L_{\infty e}$.

Definition State-Transition Matrix

The *state-transition matrix* $\Phi(t, t_0) \in \mathbb{R}^{n \times n}$ associated with $A(t)$ is, by definition, the unique solution of the matrix differential equation

$$\frac{d}{dt}(\Phi(t, t_0)) = A(t)\Phi(t, t_0) \qquad \Phi(t_0, t_0) = I \qquad (1.5.32)$$

Note that linear systems with $A(.) \in L_{\infty e}$ automatically satisfy the Lipschitz condition over any finite interval, so that the solutions of (1.5.31) and (1.5.32) are unique on any time interval. It is easy to verify that the solution of (1.5.31) is related to that of (1.5.32) through

$$x(t) = \Phi(t, t_0) x(t_0) \qquad (1.5.33)$$

In particular, this expression shows that trajectories are "proportional" to the size of the initial conditions, so that local and global properties of LTV systems are identical.

The state-transition matrix satisfies the so-called *semigroup property* (cf. Kailath [1980], p. 599):

$$\Phi(t, t_0) = \Phi(t, \tau)\Phi(\tau, t_0) \qquad \text{for all } t \geq \tau \geq t_0 \qquad (1.5.34)$$

and its inverse is given by

$$\Phi(t, t_0)^{-1} = \Phi(t_0, t) \qquad (1.5.35)$$

A consequence is that

$$\begin{aligned} \frac{d}{dt_0} \Phi(t, t_0) &= \frac{d}{dt_0} (\Phi(t_0, t))^{-1} \\ &= -\Phi(t_0, t)^{-1} A(t_0) \Phi(t_0, t) \Phi(t_0, t)^{-1} \\ &= -\Phi(t, t_0) A(t_0) \end{aligned} \qquad (1.5.36)$$

The following propositions relate the stability properties of (1.5.31) to properties of the state-transition matrix.

Proposition 1.5.3 Uniform Asymptotic Stability of LTV Systems

$x = 0$ is a uniformly asymptotically stable equilibrium point of (1.5.31) *if and only if* $x = 0$ is stable, which is guaranteed by

$$\sup_{t_0 \geq 0} \left(\sup_{t \geq t_0} \| \Phi(t, t_0) \| \right) < \infty \qquad (1.5.37)$$

and $x = 0$ is attractive, which is guaranteed by

$$\| \Phi(t, t_0) \| \to 0 \text{ as } t \to \infty \qquad \text{uniformly in } t_0. \qquad (1.5.38)$$

Proof of Proposition 1.5.3 direct from the expression of the solution (1.5.33).

Proposition 1.5.4 Exponential Stability of LTV Systems

$x = 0$ is an exponentially stable equilibrium point of (1.5.31)

if and only if for some $m, \alpha > 0$

$$\| \Phi(t,t_0) \| = m e^{-\alpha(t-t_0)} \tag{1.5.39}$$

for all $t \geq t_0 \geq 0$.

Proof of Proposition 1.5.4 direct from the expression of the solution (1.5.33).

A unique property of linear systems is the equivalence between uniform asymptotic stability and exponential stability, as stated in the following theorem.

Proposition 1.5.5 Exponential and Uniform Asymptotic Stability

$x = 0$ is a uniformly asymptotically stable equilibrium point of (1.5.31)

if and only if $x = 0$ is an exponentially stable equilibrium point of (1.5.31).

Proof of Proposition 1.5.5

That exponential stability implies uniform asymptotic stability is obvious from their definitions and in particular from proposition 1.5.3 and proposition 1.5.4. We now show the converse. Proposition 1.5.3 implies that there exists $M > 0$ and $T > 0$, such that

$$\| \Phi(t, t_0) \| < M \qquad \text{for all } t \geq t_0 \geq 0 \tag{1.5.40}$$

and

$$\| \Phi(t_0 + T, t_0) \| < \frac{1}{2} \qquad \text{for all } t_0 \geq 0 \tag{1.5.41}$$

For all $t \geq t_0$, there exists an integer n such that $t \in [t_0 + nT, t_0 + (n+1)T]$. Using the semigroup property recursively, together with (1.5.40), (1.5.41)

$$\| \Phi(t,t_0) \| \leq \| \Phi(t, t_0 + nT) \| \ \| \Phi(t_0 + nT, t_0) \|$$

$$< M \left[\frac{1}{2}\right]^n \leq M \left[\frac{1}{2}\right]^{(t-t_0)/T} \tag{1.5.42}$$

which can easily be expressed in the form of (1.5.39). □

Uniform Complete Observability–Definition and Results

Through the definition of uniform complete observability, some additional results on the stability of linear time-varying systems will now be established. We consider the linear time-varying system $[C(t), A(t)]$ defined by

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) \\ y(t) &= C(t)x(t) \end{aligned} \tag{1.5.43}$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}^m$, while $A(t) \in \mathbb{R}^{n \times n}$, $C(t) \in \mathbb{R}^{m \times n}$, are piecewise continuous functions (therefore belonging to $L_{\infty e}$).

Definition Uniform Complete Observability (UCO)

The system $[C(t), A(t)]$ is called *uniformly completely observable* (UCO) if there exist strictly positive constants β_1, β_2, δ, such that, for all $t_0 \geq 0$

$$\beta_2 I \geq N(t_0, t_0 + \delta) \geq \beta_1 I \tag{1.5.44}$$

where $N(t_0, t_0 + \delta) \in \mathbb{R}^{n \times n}$ is the so-called *observability grammian*

$$N(t_0, t_0 + \delta) = \int_{t_0}^{t_0+\delta} \Phi^T(\tau, t_0) C^T(\tau) C(\tau) \Phi(\tau, t_0) \, d\tau \tag{1.5.45}$$

Comments

Note that, using (1.5.33), condition (1.5.44) can be rewritten as

$$\beta_2 |x(t_0)|^2 \geq \int_{t_0}^{t_0+\delta} |C(\tau)x(\tau)|^2 d\tau \geq \beta_1 |x(t_0)|^2 \tag{1.5.46}$$

for all $x(t_0) \in \mathbb{R}^n$, $t_0 \geq 0$, where $x(t)$ is the solution of (1.5.43) starting at $x(t_0)$.

The observability is called uniform because (1.5.43) is satisfied uniformly for all t_0 and complete because (1.5.46) is satisfied for all $x(t_0)$. A specific $x(t_0)$ is observable on a specific time interval $[t_0, t_0 + \delta]$ if condition (1.5.46) is satisfied on that interval. Then, $x(t_0)$ can be reconstructed from the knowledge of $y(.)$ using the expression

$$x(t_0) = N(t_0, t_0 + \delta)^{-1} \int_{t_0}^{t_0+\delta} \Phi^T(\tau, t_0) C^T(\tau) y(\tau) \, d\tau \tag{1.5.47}$$

On the other hand, if condition (1.5.44) fails to be satisfied, (1.5.46) shows that there exists an $x(t_0) \neq 0$ such that the corresponding output satisfies

$$\int_{t_0}^{t_0+\delta} |y(\tau)|^2 d\tau = 0 \tag{1.5.48}$$

so that $x(t_0)$ is not distinguishable from 0.

Theorem 1.5.6 Exponential Stability of LTV Systems
The following statements are equivalent:

(a) $x = 0$ is an exponentially stable equilibrium point of (1.5.31).

(b) For all $C(t) \in \mathbb{R}^{m \times n}$ (with m arbitrary) such that the pair $[C(t), A(t)]$ is UCO, there exists a symmetric $P(t) \in \mathbb{R}^{n \times n}$, and some $\gamma_1, \gamma_2 > 0$, such that

$$\gamma_2 I \geq P(t) \geq \gamma_1 I \tag{1.5.49}$$

$$-\dot{P}(t) = A^T(t)P(t) + P(t)A(t) + C^T(t)C(t) \tag{1.5.50}$$

for all $t \geq 0$.

(c) For some $C(t) \in \mathbb{R}^{m \times n}$ (with m arbitrary) such that the pair $[C(t), A(t)]$ is UCO, there exists a symmetric $P(t) \in \mathbb{R}^{n \times n}$ and some $\gamma_1, \gamma_2 > 0$, such that (1.5.49) and (1.5.50) are satisfied.

Proof of Theorem 1.5.6

(a) implies (b)

Define

$$P(t) = \int_t^{\infty} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau \tag{1.5.51}$$

We will show that (1.5.49) is satisfied so that $P(t)$ is well defined, but first note that by differentiating (1.5.51), and using (1.5.36), it follows that $P(t)$ satisfies the linear differential equation (1.5.50). This equation is a linear differential equation, so that the solution is unique for given initial conditions. However, we did not impose initial conditions in (1.5.50), and $P(0)$ is in fact given by (1.5.51).

By the UCO assumption

$$\beta_2 I \geq \int_t^{t+\delta} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau \geq \beta_1 I \tag{1.5.52}$$

for all $t \geq 0$. Since the integral from t to ∞ is not less than from t to $t + \delta$, the lower inequality in (1.5.50) follows directly from (1.5.52). To

show the upper inequality, we divide the interval of integration in intervals of size δ and sum the individual integrals. On the interval $[t + \delta, t + 2\delta]$

$$\int_{t+\delta}^{t+2\delta} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau = \Phi^T(t+\delta, t) \cdot \left[\int_{t+\delta}^{t+2\delta} \Phi^T(\tau, t+\delta) C^T(\tau) C(\tau) \Phi(\tau, t+\delta) d\tau \right] \Phi(t+\delta, t)$$

$$\leq \beta_2 m^2 e^{-2\alpha\delta} I \tag{1.5.53}$$

where we used the UCO and exponential stability assumptions. Therefore

$$\int_t^{\infty} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau$$

$$\leq \beta_2 (1 + m^2 e^{-2\alpha\delta} + m^2 e^{-4\alpha\delta} + \cdots) I := \gamma_2 I \tag{1.5.54}$$

(b) implies (c) trivial.

(c) implies (a).

Consider the Lyapunov function

$$v(t, x) = x^T(t) P(t) x(t) \tag{1.5.55}$$

so that

$$-\dot{v} = -x^T(t)P(t)A(t)x(t) - x^T(t)A^T(t)P(t)x(t) - x^T(t)\dot{P}(t)x(t) = x^T(t)C^T(t)C(t)x(t) \geq 0 \tag{1.5.56}$$

Using the UCO property

$$\int_t^{t+\delta} \dot{v} \, d\tau = -x^T(t) \left[\int_t^{t+\delta} \Phi^T(\tau, t) C^T(\tau) C(\tau) \Phi(\tau, t) d\tau \right] x(t)$$

$$\leq -\beta_1 |x(t)|^2 \tag{1.5.57}$$

Exponential convergence follows from theorem 1.5.2.

It is interesting to note that the Lyapunov function is identical to the function used in theorem 1.5.1, when $C = I$, and the upper bound of integration tends to infinity. □

1.5.3 Exponential Stability of Linear Time Invariant Systems

When the system (1.5.31) is time-invariant, even further simplification of the stability criterion results. Specifically, we consider the system

$$\dot{x} = A x \qquad x(t_0) = x_0 \tag{1.5.58}$$

Note that since the system is linear, local and global stability properties are identical, and since it is time-invariant, all properties are uniform (in t_0).

Theorem 1.5.7 Lyapunov Lemma

The following statements are equivalent:

(a) All eigenvalues of A lie in the open left-half plane

(b) $x = 0$ is an exponentially stable equilibrium point of (1.5.58)

(c) For all $C \in \mathbb{R}^{m \times n}$ (with m arbitrary), such that the pair $[C, A]$ is observable, there exists a symmetric positive definite $P \in \mathbb{R}^{n \times n}$ satisfying

$$A^T P + PA = -C^T C \tag{1.5.59}$$

(d) For *some* $C \in \mathbb{R}^{m \times n}$ (with m arbitrary), such that the pair $[C, A]$ is observable, there exists a symmetric positive definite $P \in \mathbb{R}^{n \times n}$ satisfying (1.5.59).

Proof of Theorem 1.5.7

(a) implies (b).

This follows from the fact that the transition matrix is given by the exponential matrix (cf. Vidyasagar [1978], pp. 171)

$$\Phi(t, t_0) = e^{A(t - t_0)} \tag{1.5.60}$$

(b) implies (c).

Let

$$S(t) := \int_0^t e^{A^T \tau} C^T C e^{A\tau} d\tau$$

$$= \int_0^t e^{A^T (t-\sigma)} C^T C e^{A(t-\sigma)} d\sigma \tag{1.5.61}$$

Clearly $S(t) = S^T(t)$, and since $[C, A]$ is observable, $S(t) > 0$ for all $t > 0$. Using both expressions in (1.5.61)

$$\dot{S}(t) = e^{A^T t} C^T C e^{At} = A^T S(t) + S(t) A + C^T C \tag{1.5.62}$$

and using the exponential stability of A

$$\lim_{t \to \infty} \dot{S}(t) = 0 = A^T P + PA + C^T C \tag{1.5.63}$$

where

$$P = \lim_{t \to \infty} S(t) \tag{1.5.64}$$

(c) implies (d) trivial.

(d) implies (b) as in theorem 1.5.6. □

Remark

The original version of the Lyapunov lemma is stated with Q positive definite replacing $C^T C$ (which is only semi-definite), and leading to the so-called *Lyapunov equation*:

$$A^T P + PA = -Q \tag{1.5.65}$$

Then, the observability hypothesis is trivial. We stated a more general version of the Lyapunov lemma partly to show the intimate connection between exponential stability and observability for linear systems.

1.6 GENERALIZED HARMONIC ANALYSIS

An appropriate tool for the study of stable linear time-invariant systems is Fourier transforms or harmonic analysis, since $e^{j\omega t}$ is an eigenfunction of such systems. We will be concerned with the analysis of algorithms for identification and adaptive control of linear time-invariant systems. The overall adaptive systems are time-varying systems, but, under certain conditions, they become "asymptotically" time-invariant. For these systems, generalized harmonic analysis is a useful tool of analysis. The theory has been known since the early part of the century and was developed in Wiener's *Generalized Harmonic Analysis* (Wiener [1930]). Boyd & Sastry [1983] and [1986] illustrated its application to adaptive systems. Since the proofs of the various lemmas used in this book are neither difficult nor long, we provide them in this section.

Definition Stationarity, Autocovariance

A signal $u : \mathbb{R}_+ \to \mathbb{R}^n$ is said to be *stationary* if the following limit exists, uniformly in t_0

$$R_u(t) := \lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} u(\tau) u^T(t+\tau)\, d\tau \quad \in \mathbb{R}^{n\times n} \tag{1.6.1}$$

in which instance, the limit $R_u(t)$ is called the *autocovariance* of u.

The concept of autocovariance is well known in the theory of stochastic systems. There is a strong analogy between (1.6.1) and $R_u^{stoch}(t) = \mathrm{E}\,[u(\tau)u^T(t+\tau)]$, when u is a wide sense stationary stochastic process. Indeed, for a wide sense stationary *ergodic* process $u(t,\omega)$ (ω here denotes a sample point of the underlying probability space), the autocovariance $R_u(t,\omega)$ exists, and is equal to $R_u^{stoch}(t)$ for almost all ω. But we emphasize that the autocovariance defined in (1.6.1) is completely deterministic.

Proposition 1.6.1

R_u is a *positive semi-definite matrix-valued function,* that is, for all $t_1, \ldots, t_k \in \mathbb{R}$, and $c_1, \ldots, c_k \in \mathbb{R}^n$

$$\sum_{i,j=1}^{k} c_i^T R_u(t_j - t_i) c_j \ \geq\ 0 \tag{1.6.2}$$

Proof of Proposition 1.6.1

Define the scalar valued function $v(t)$ by

$$v(t) := \sum_{i=1}^{k} c_i^T u(t+t_i) \tag{1.6.3}$$

Then, for all $T > 0$

$$0 \leq \frac{1}{T}\int_0^T |v(\tau)|^2 d\tau \tag{1.6.4}$$

$$= \sum_{i,j=1}^{k} c_i^T \Big(\frac{1}{T}\int_0^T u(\tau+t_i) u^T(\tau+t_j)\, d\tau\Big) c_j$$

$$= \sum_{i,j=1}^{k} c_i^T \Big(\frac{1}{T}\int_{t_i}^{t_i+T} u(\sigma) u^T(\sigma+t_j-t_i)\, d\sigma\Big) c_j \tag{1.6.5}$$

Since u has an autocovariance, the limit of (1.6.5) as $T \to \infty$ is

$$\sum_{i,j=1}^{k} c_i^T R_u(t_j - t_i) c_j \tag{1.6.6}$$

From (1.6.4), we find that (1.6.6) is positive, so that (1.6.2) follows. □

From proposition 1.6.1, it follows (see, for example, Widder [1971]) that R_u is the Fourier transform of a positive semi-definite matrix $S_u(d\omega)$ of bounded measures, that is,

$$R_u(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{j\omega t} S_u(d\omega) \tag{1.6.7}$$

and

$$\int_{-\infty}^{\infty} S_u(d\omega) = 2\pi R_u(0) < \infty \tag{1.6.8}$$

The corresponding inverse transform is

$$S_u(d\omega) = \int_{-\infty}^{+\infty} e^{-j\omega\tau} R_u(\tau)\, d\tau \tag{1.6.9}$$

The matrix measure $S_u(d\omega)$ is referred to as the *spectral measure* of the signal u. In particular, if u has a sinusoidal component at frequency ω_0, then u is said to have a *spectral line* at frequency ω_0. $S_u(d\omega)$ has point mass (a delta function) at ω_0 and $-\omega_0$. If $u(t)$ is periodic, with period $2\pi/\omega_0$, then $S_u(d\omega)$ has point mass at ω_0, $2\omega_0, \ldots$ and $-\omega_0$, $-2\omega_0, \ldots$. Further, if $u(t)$ is almost periodic, its spectral measure has point masses concentrated at countably many points.

Since R_u is real, (1.6.9) shows that $\mathrm{Re}(S_u)$ is an even function of ω, and $\mathrm{Im}(S_u)$ is an odd function of ω. On the other hand, (1.6.1) shows that

$$R_u(t) = R_u^T(-t)$$

Therefore, (1.6.9) also shows that $\mathrm{Re}(S_u)$ is a symmetric matrix, while $\mathrm{Im}(S_u)$ is an antisymmetric matrix. In other words,

$$S_u^T(d\omega) = S_u^*(d\omega)$$

Equations (1.6.1) and (1.6.9) give a technique for obtaining the spectral content of stationary deterministic signals. For example, consider the scalar signal

$$u(t) = \sin(\omega_1 t) + \sin(\omega_2 t) + f(t) \tag{1.6.10}$$

where $f(t)$ is some continuous function that tends to zero as $t \to \infty$. The signal u has autocovariance

$$R_u(t) = \frac{1}{2}\cos(\omega_1 t) + \frac{1}{2}\cos(\omega_2 t) \tag{1.6.11}$$

showing spectral content at ω_1 and ω_2.

The autocovariance of the input and of the output signals of a stable linear time-invariant system can be related as in the following proposition.

Proposition 1.6.2 Linear Filter Lemma

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper, stable $m \times n$ matrix transfer function, with real impulse response $H(t)$.

If u is stationary, with autocovariance $R_u(t)$

Then y is stationary, with autocovariance

$$R_y(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H(\tau_1) R_u(t + \tau_1 - \tau_2) H^T(\tau_2)\, d\tau_1\, d\tau_2 \tag{1.6.12}$$

and spectral measure

$$S_y(d\omega) = \hat{H}^*(j\omega) S_u(d\omega) \hat{H}^T(j\omega) \tag{1.6.13}$$

Proof of Proposition 1.6.2

We first establish that y has an autocovariance, by considering

$$\frac{1}{T}\int_{t_0}^{t_0+T} y(\tau) y^T(t+\tau)\, d\tau = \frac{1}{T}\int_{t_0}^{t_0+T} \left(\int H(\tau_1) u(\tau - \tau_1)\, d\tau_1\right) \cdot \left(\int u^T(t + \tau - \tau_2) H^T(\tau_2)\, d\tau_2\right) d\tau \tag{1.6.14}$$

For all T, the integral in (1.6.14) exists absolutely, since $\hat{H}$ is stable, and therefore $H \in L_1$. Therefore, we may change the order of integration, to obtain

$$\iint H(\tau_1)\, d\tau_1 \left[\frac{1}{T}\int_{t_0-\tau_1}^{t_0-\tau_1+T} u(\sigma) u^T(t + \sigma + \tau_1 - \tau_2)\, d\sigma\right] (H^T(\tau_2)\, d\tau_2) \tag{1.6.15}$$

The expression in parenthesis converges to $R_u(t + \tau_1 - \tau_2)$ as $T \to \infty$, uniformly in t_0. Further, it is bounded as a function of T, t_0, τ_1 and τ_2, since, by the Schwarz inequality

$$\left|\frac{1}{T}\int_{t_0-\tau_1}^{t_0-\tau_1+T} u(\sigma) u^T(t + \sigma + \tau_1 - \tau_2)\, d\sigma\right| \le \sup_{t_0, T} \frac{1}{T}\int_{t_0}^{t_0+T} |u(\tau)|^2\, d\tau \tag{1.6.16}$$

Hence, by dominated convergence, (1.6.15) converges uniformly in t_0, as $T \to \infty$, to (1.6.12), so that y has an autocovariance given by (1.6.12).

To complete the proof, we substitute (1.6.7) in (1.6.12) to get

$$\begin{aligned} R_y(t) &= \frac{1}{2\pi}\iint H(\tau_1)\, d\tau_1 \int e^{j\omega(t+\tau_1-\tau_2)} S_u(d\omega) H^T(\tau_2)\, d\tau_2 \\ &= \frac{1}{2\pi}\int e^{j\omega t}\left(\int e^{j\omega\tau_1} H(\tau_1)\, d\tau_1\right) S_u(d\omega) \left(\int e^{-j\omega\tau_2} H(\tau_2)\, d\tau_2\right)^T \\ &= \frac{1}{2\pi}\int e^{j\omega t} \hat{H}(-j\omega) S_u(d\omega) \hat{H}^T(j\omega) \end{aligned} \tag{1.6.17}$$

Note that (1.6.17) is the Fourier representation of R_y, so that, using the fact that $H(t)$ is real

$$\begin{aligned} S_y(d\omega) &= \hat{H}(-j\omega) S_u(d\omega) \hat{H}^T(j\omega) \\ &= \hat{H}^*(j\omega) S_u(d\omega) \hat{H}^T(j\omega) \end{aligned} \tag{1.6.18}$$

□

Remark

It is easy to see that if u has a spectral line at frequency ω_0, so does y, and the intensity of the spectral line of y at ω_0 is given by (1.6.18). Note, however, that if $\hat{H}(s)$ has a zero of transmission at ω_0, then the amplitude of the spectral line at the output is zero.

We will also need to define the cross correlation and cross spectral density between two stationary signals.

Definition Cross Correlation

Let $u: \mathbb{R}_+ \to \mathbb{R}^n$ and $y: \mathbb{R}_+ \to \mathbb{R}^m$ be stationary. The *cross correlation* between u and y is defined to be the following limit, uniform in t_0

$$R_{yu}(t) := \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} y(\tau)\, u^T(t+\tau)\, d\tau \in \mathbb{R}^{m \times n} \quad (1.6.19)$$

It may be verified that the relationship between R_{uy} and R_{yu} is

$$R_{yu}(t) = R_{uy}^T(-t) \quad (1.6.20)$$

and the Fourier transform of the cross correlation is referred to as the *cross spectral measure* of u and y.

The cross correlation between the input and the output of a stable LTI system can be related in much the same way as in proposition 1.6.2.

Proposition 1.6.3 Linear Filter Lemma—Cross Correlation

Let $y = \hat{H}(u)$, where $\hat{H}$ is a proper stable $m \times n$ matrix transfer function, with impulse response $H(t)$.

If u is stationary

Then the cross correlation between u and y is given by

$$R_{yu}(t) = \int_{-\infty}^{\infty} H(\tau_1) R_u(t+\tau_1)\, d\tau_1 \quad (1.6.21)$$

and the cross spectral measure is

$$S_{yu}(d\omega) = \hat{H}^*(j\omega)\, S_u(d\omega) \quad (1.6.22)$$

Proof of Proposition 1.6.3

The proof is analogous to the proof of proposition 1.6.2 and is omitted here.

CHAPTER 2
IDENTIFICATION

2.0 INTRODUCTION

In this chapter, we review some identification methods for single-input single-output (SISO), linear time invariant (LTI) systems. To introduce the subject, we first informally discuss a simple example. We consider the identification problem for a first order SISO LTI system described by a transfer function

$$\frac{\hat{y}_p(s)}{\hat{r}(s)} = \hat{P}(s) = \frac{k_p}{s + a_p} \quad (2.0.1)$$

The parameters k_p and a_p are unknown and are to be determined by the identification scheme on the basis of measurements of the input and output of the plant. The plant is assumed to be stable, i.e. $a_p > 0$.

Frequency Domain Approach

A standard approach to identification is the *frequency response* approach. Let the input r be a sinusoid

$$r(t) = \sin(\omega_0 t) \quad (2.0.2)$$

The steady-state response is then given by

$$y(t) = m \sin(\omega_0 t + \phi) \quad (2.0.3)$$

where