the convergence rates, even in the nonlinear adaptive control case. These results are useful for the optimum design of reference input. They have the limitation of depending on unknown plant parameters, but an approximation of the complete parameter trajectory is obtained and the understanding of the dynamical behavior of the parameter error is considerably increased using averaging. For example, it was found that the trajectory of the parameter error corresponding to the linear error equation could be approximated by an LTI system with real negative eigenvalues, while for the strictly positive real (SPR) error equation it had possibly complex eigenvalues.

Besides requiring stationarity of input signals, averaging also required slow parameter adaptation. We showed however, through simulations, that the approximation by the averaged system was good for values of the adaptation gain that were close to 1 (that is, not necessarily infinitesimal) and for acceptable time constants in the parameter variations. In fact, it appeared that a basic condition is simply that parameters vary more slowly than do other states and signals of the adaptive system.

CHAPTER 5
ROBUSTNESS

5.1 STRUCTURED AND UNSTRUCTURED UNCERTAINTY

In a large number of control system design problems, the designer does not have a detailed state-space model of the plant to be controlled, either because it is too complex, or because its dynamics are not completely understood. Even if a detailed high-order model of the plant is available, it is usually desirable to obtain a reduced order controller, so that part of the plant dynamics must be neglected. We begin discussing the representation of such uncertainties in plant models, in a framework similar to Doyle & Stein [1981].

Consider the kind of prior information available to control a *stable* plant, and obtained for example by performing input-output experiments, such as sinusoidal inputs. Typically, Bode diagrams of the form shown in Figures 5.1 and 5.2 are obtained. An inspection of the diagrams shows that the data obtained beyond a certain frequency ω_H is unreliable because the measurements are poor, corrupted by noise, and so on. They may also correspond to the high-order dynamics that one wishes to neglect. What is available, then, is essentially no phase information, and only an "envelope" of the magnitude response beyond ω_H. The dashed lines in the magnitude and phase response correspond to the approximation of the plant by a finite order model, assuming that there are no dynamics at frequencies beyond ω_H. For frequencies below ω_H, it is easy to guess the presence of a zero near ω_1, poles in the neighborhood of ω_2, ω_3, and complex pole pairs in the neighborhood of ω_4, ω_5.

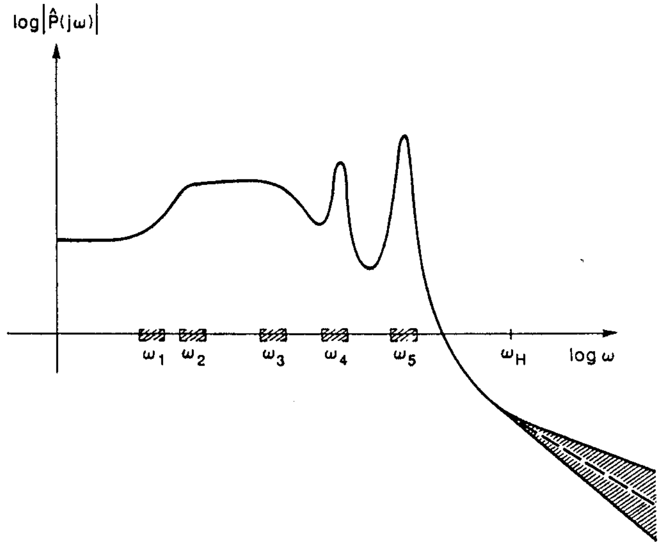

Figure 5.1: Bode Plot of the Plant (Gain)

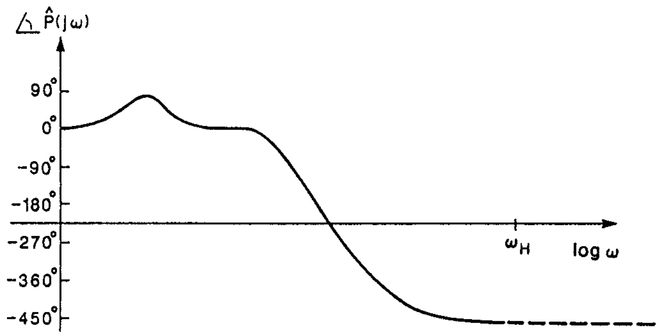

Figure 5.2: Bode Plot of the Plant (Phase)

To keep the design goal specific and consistent with our previous analysis, we will assume that the designer's goal is *model following*: the designer is furnished with a desired closed-loop response and selects an appropriate reference model with transfer function $\hat{M}(s)$. The problem is to design a control system to get the plant output $y_p(t)$ to track the

model output $y_m(t)$ in response to reference signals $r(t)$ driving the model. This is shown in Figure 5.3.

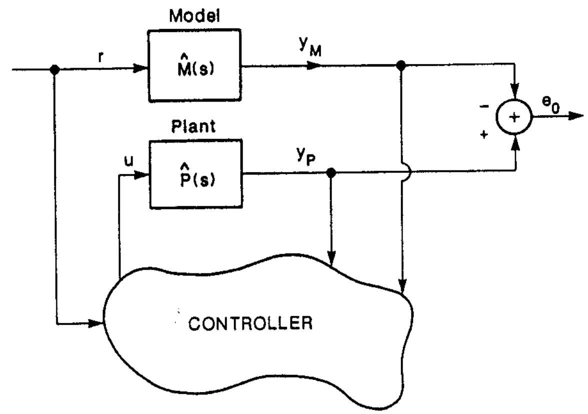

Figure 5.3: Model Following Control System

The controller generates the input $u(t)$ of the plant, using $y_m(t)$, $y_p(t)$ and $r(t)$ so that the error between the plant and model output $e_0(t) := y_p(t) - y_m(t)$ tends to zero asymptotically.

Two options are available to the designer at this point.

Non-Adaptive Robust Control. The designer uses as model for the plant the nominal transfer function $\hat{P}^*(s)$

$$\hat{P}^*(s) = \frac{k_p(s+\omega_1)}{(s+\omega_2)\,(s+\omega_3)((s+\nu_4)^2+(\omega_4)^2)((s+\nu_5)^2+(\omega_5)^2)} \tag{5.1.1}$$

The gain k_p in (5.1.1) is obtained from the nominal high-frequency asymptote of Figure 5.1 (i.e. the dashed line). The modeling errors due to inaccuracies in the pole-zero locations, and to poor data at high frequencies may be taken into account by assuming that the actual plant transfer function is of the form

$$\hat{P}(s) = \hat{P}^*(s) + \hat{H}_a(s) \tag{5.1.2}$$

or

$$\hat{P}(s) = \hat{P}^*(s)\,(1+\hat{H}_m(s)) \tag{5.1.3}$$

where $\hat{H}_a(s)$ is referred to as the *additive uncertainty* and $\hat{H}_m(s)$ as the *multiplicative uncertainty*. Of course, $|\hat{H}_a(j\omega)|$ and $|\hat{H}_m(j\omega)|$ are unknown, but magnitude bounds may be determined from input-output measurements and other available information. A typical bound for $|\hat{H}_m(j\omega)|$ is shown in Figure 5.4.

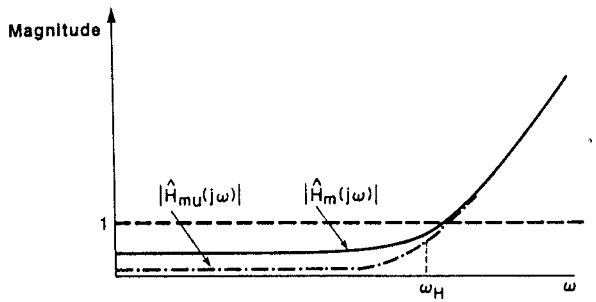

Figure 5.4: Typical Plot of Uncertainty $|\hat{H}_m(j\omega)|$ and $|\hat{H}_{mu}(j\omega)|$

Given the desired transfer function $\hat{M}(s)$, one attempts to build a *linear, time-invariant* controller of the form shown in Figure 5.5, with feedforward compensator $\hat{C}(s)$ and feedback compensator $\hat{F}(s)$, so that the nominal closed-loop transfer function approximately matches the reference model, that is,

$$\hat{P}^*(s)\hat{C}(s)\left[I + \hat{F}(s)\hat{P}^*(s)\hat{C}(s)\right]^{-1} \sim \hat{M}(s) \qquad (5.1.4)$$

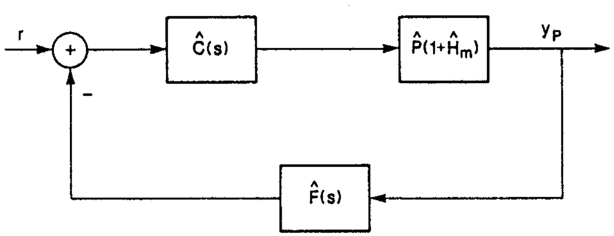

Figure 5.5: Non-adaptive Controller Structure

over the frequency range of interest (the frequency range of r). Further, $\hat{C}(s)$ and $\hat{F}(s)$ are chosen so as to at least *preserve stability* and also *reduce sensitivity* of the actual closed-loop transfer function to the modeling errors represented by $\hat{H}_a$, or $\hat{H}_m$ within some given bounds.

Adaptive Control. The designer makes a distinction between the two kinds of uncertainty present in the description of Figures 5.1–5.2: the *parametric* or *structured* uncertainty in the pole and zero locations and the *inherent* or *unstructured* uncertainty due to additional dynamics beyond ω_H. Rather than postulate a transfer function for the plant, the designer decides to identify the pole-zero locations *on-line*, i.e. during the operation of the plant. This on-line "tune-up" is for the purpose of reduction of the structured uncertainty during the course of plant operation. The aim is to obtain a better match between $\hat{M}(s)$ and the controlled plant for frequencies below ω_H. A key feature of the on-line tuning approach is that the controller is generally nonlinear and time-varying. The added complexity of adaptive control is made worthwhile when the performance achieved by non-adaptive control is inadequate.

The plant model for adaptive control is given by

$$\hat{P}(s) = \hat{P}_{\theta^*}(s) + \hat{H}_{au}(s) \qquad (5.1.5)$$

or

$$\hat{P}(s) = \hat{P}_{\theta^*}(s)(1 + \hat{H}_{mu}(s)) \qquad (5.1.6)$$

where $\hat{P}_{\theta^*}(s)$ stands for the plant indexed by the parameters θ^* and $\hat{H}_{au}(s)$ and $\hat{H}_{mu}(s)$ are the additive and multiplicative uncertainties respectively. The difference between (5.1.2)–(5.1.3) and (5.1.5)–(5.1.6) lies in the on-line tuning of the parameter θ^* to reduce the uncertainty, so that it only consists of the *unstructured uncertainty* due to high-frequency unmodeled dynamics.

When the plant is *unstable*, a frequency response curve as shown in Figures 5.1–5.2 is not available, and a certain amount of off-line identification and detailed modeling needs to be performed. As before, however, the plant model will have both *structured* and *unstructured* uncertainty, and the design options will be the same as above. The difference only arises in the representation of uncertainty. Consider, for example, the multiplicative uncertainty in the nonadaptive and adaptive cases. Previously, $\hat{H}_m(s)$ was stable. However, when the plant is unstable, since the nominal locations of the unstable poles may not be chosen exactly, $\hat{H}_m(s)$ may be an unstable transfer function. For adaptive control, we require merely that *all unstable poles of the system be parameterized* (of course, their exact location is not essential!), so that the description for the uncertainty is still given by (5.1.6), with $\hat{H}_{mu}(s)$ stable, even though $\hat{P}_{\theta^*}(s)$ may not be.

A simple example illustrates this: consider a plant with transfer function

$$\hat{P}(s) = \frac{m}{(s - 1 + \epsilon)(s + m)} \tag{5.1.7}$$

with $\epsilon > 0$ small and $m > 0$ large.

For non-adaptive control, the nominal plant is chosen to be $1/s - 1$, so that

$$\hat{H}_{mu}(s) = \frac{-s^2 + s - \epsilon(s + m)}{(s - 1 + \epsilon)(s + m)} \quad \text{(unstable)} \tag{5.1.8}$$

For adaptive control on the other hand, $\hat{P}_{\theta^*}(s) = 1/(s + \theta^*)$ is chosen with

$$\hat{H}_{mu}(s) = -\frac{s}{s + m} \quad \text{(stable)} \tag{5.1.9}$$

and $\theta^* = -1 + \epsilon$.

In the preceding chapters, we only considered the adaptive control of plants with parameterized uncertainty, i.e., control of $\hat{P}_{\theta^*}$. Specifically, we choose $\hat{P}_{\theta^*}$ of the form $k_p \hat{n}_p / \hat{d}_p$, where $\hat{n}_p$, $\hat{d}_p$ are monic, coprime polynomials of degrees m, n respectively. We assumed that

(a) The number of poles of $\hat{P}_{\theta^*}$, that is, n, is known.

(b) The number of zeros of $\hat{P}_{\theta^*}$, that is, $m \leq n$, is known.

(c) The sign of the high-frequency gain k_p is known (a bound may also be required) .

(d) $\hat{P}_{\theta^*}$ is minimum phase, that is, the zeros of $\hat{n}_p$ lie in the open left half plane (LHP).

It is important to note that the assumptions apply to the *nominal* plant $\hat{P}_{\theta^*}$. In particular, $\hat{P}$ may have many more stable poles and zeros than $\hat{P}_{\theta^*}$. Further, the sign of the high-frequency gain of $\hat{P}$ is usually indeterminate as shown in Figure 5.1.

The question is, of course,: how will the adaptive algorithms described in previous chapters behave with the true plant $\hat{P}$? A basic desirable property of the control algorithm is to maintain stability in the presence of uncertainties. This property is usually referred to as the *robustness* of the control algorithm.

A major difficulty in the definition of robustness is that it is very problem dependent. Clearly, an algorithm which could not tolerate *any* uncertainty (that is, no matter how small) would be called non robust. However, it would also be considered non robust in practice, if the range of tolerable uncertainties were smaller than the actual uncertainties present in the system. Similarly, an algorithm may be sufficiently robust for one application, and not for another. A key set of observations made by Rohrs, Athans, Valavani & Stein [1982, 1985] is that adaptive control algorithms which are proved stable by the techniques of previous chapters can become unstable in the presence of mild unmodeled dynamics or arbitrarily small output disturbances. We start by reviewing their examples.

5.2 THE ROHRS EXAMPLES

Despite the existence of stability proofs for adaptive control systems (cf. Chapter 3), Rohrs *et al* [1982], [1985] showed that several algorithms can become unstable when some of the assumptions required by the stability proofs are not satisfied. While Rohrs (we drop the *et al* for compactness) considered several continuous and discrete time algorithms, the results are qualitatively similar for the various schemes. We consider one of these schemes here, which is the output error direct adaptive control scheme of Section 3.3.2, assuming that the degree and the relative degree of the plant are 1.

The adaptive control scheme of Rohrs examples is designed assuming a first order plant with transfer function

$$\hat{P}_{\theta^*}(s) = \frac{k_p}{s + a_p} \tag{5.2.1}$$

and the strictly positive real (SPR) reference model

$$\hat{M}(s) = \frac{k_m}{s + a_m} = \frac{3}{s + 3} \tag{5.2.2}$$

The output error adaptive control scheme (cf. Section 3.3.2) is described by

$$u = c_0 r + d_0 y_p \tag{5.2.3}$$

$$e_0 = y_p - y_m \tag{5.2.4}$$

$$\dot{c}_0 = -g\, e_0 r \tag{5.2.5}$$

$$\dot{d}_0 = -g\, e_0 y_p \tag{5.2.6}$$

As a first step, we assume that the plant transfer function is given by (5.2.1), with $k_p = 2$, $a_p = 1$. The nominal values of the controller parameters are then

$$c_0^* = \frac{k_m}{k_p} = 1.5 \tag{5.2.7}$$

$$d_0^* = \frac{a_p - a_m}{k_p} = -1 \tag{5.2.8}$$

The behavior of the adaptive system is then studied, assuming that the *actual* plant does not satisfy exactly the assumptions on which the adaptive control system is based. The actual plant is only *approximately* a first order plant and has the third order transfer function

$$\hat{P}(s) = \frac{2}{s+1} \cdot \frac{229}{s^2 + 30s + 229} \tag{5.2.9}$$

In analogy with nonadaptive control terminology, the second term is called the *unmodeled dynamics*. The poles of the unmodeled dynamics are located at $-15 \pm j2$, and, at low frequencies, this term is approximately equal to 1.

In Rohrs examples, the measured output $y_p(t)$ is also affected by a measurement noise $n(t)$. The actual plant with the reference model and the controller are shown in Figure 5.6.

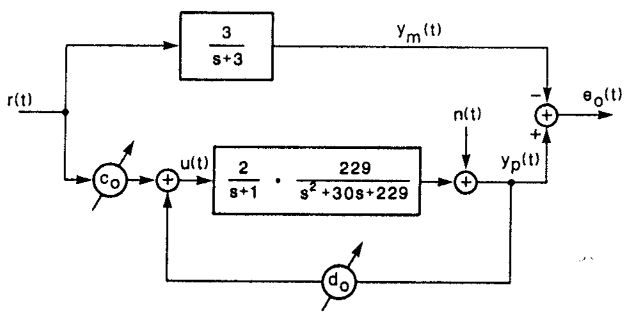

Figure 5.6: Rohrs Example—Plant, Reference Model, and Controller

An important aspect of Rohrs examples is that the modes of the actual plant and those of the model are well within the stability region. Moreover, the unmodeled dynamics are well-damped, stable modes. From a traditional control design standpoint, they would be considered rather innocuous.

At the outset, Rohrs showed through simulations that, without measurement noise or unmodeled dynamics, the adaptive scheme is stable and the output error converges to zero, as predicted by the stability analysis.

However, *with unmodeled dynamics*, three different mechanisms of instability appear:

(R1) With a *large, constant* reference input and no measurement noise, the output error initially converges to zero, but eventually diverges to infinity, along with the controller parameters c_0 and d_0.

Figures 5.7 and 5.8 show a simulation with $r(t) = 4.3$, $n(t) = 0$, that illustrates this behavior ($c_0(0) = 1.14$, $d_0(0) = -0.65$ and other initial conditions are zero).

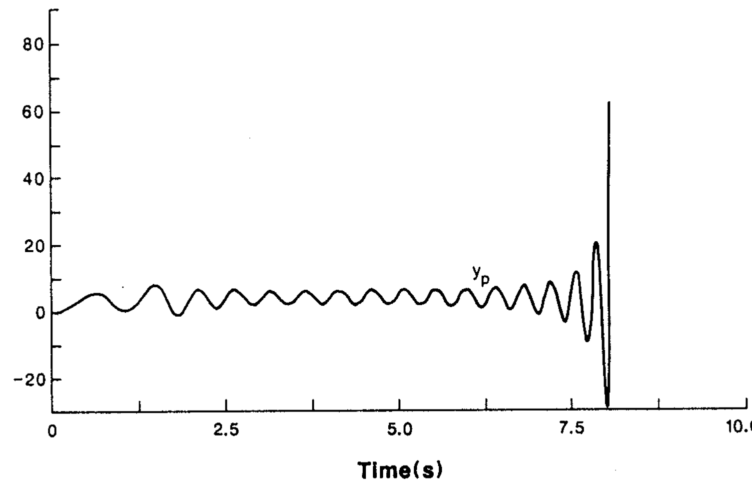

Figure 5.7 Plant Output ($r = 4.3$, $n = 0$)

(R2) With a reference input having a *small constant* component and a *large high frequency* component, the output error diverges at first slowly, and then more rapidly to infinity, along with the controller parameters c_0 and d_0.

Figures 5.9 and 5.10 show a simulation with $r(t) = 0.3 + 1.85 \sin 16.1t$, $n(t) = 0$ ($c_0(0) = 1.14$, $d_0(0) = -0.65$, and other initial conditions are zero).

(R3) With a moderate *constant input* and a small *output disturbance*, the output error initially converges to zero. After staying in the neighborhood of zero for an extended period of time, it diverges to infinity. On the other hand, the controller parameters c_0 and d_0 drift apparently at a constant rate, until they suddenly diverge to infinity.

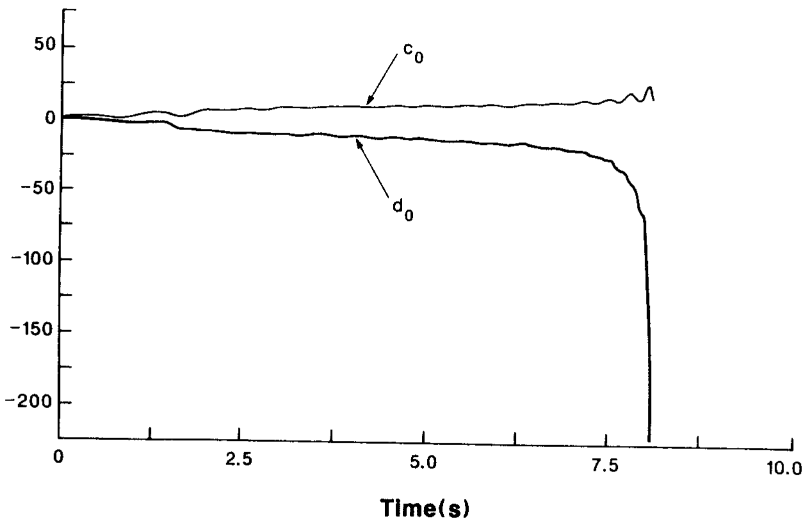

Figure 5.8 Controller Parameters ($r = 4.3$, $n = 0$)

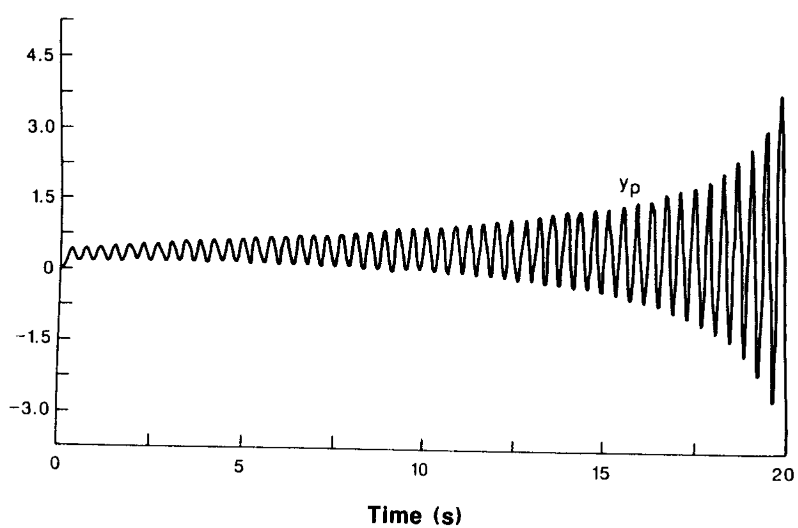

Figure 5.9 Plant Output ($r = 0.3 + 1.85 \sin 16.1t$, $n = 0$)

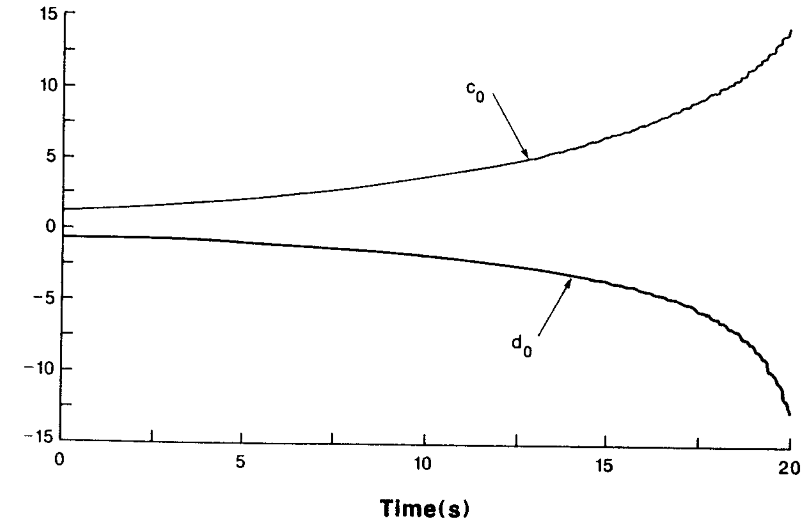

Figure 5.10 Controller Parameters ($r = 0.3 + 1.85 \sin 16.1t$, $n = 0$)

Figures 5.11 and 5.12 show a simulation with $r(t) = 2$, $n(t) = 0.5 \sin 16.1t$ ($c_0(0) = 1.14$, $d_0(0) = -0.65$, and other initial conditions are zero).

Although this simulation corresponds to a comparatively high value of $n(t)$, simulations show that when smaller values of the output disturbance $n(t)$ are present, instability still appears, but after a longer period of time. The controller parameters simply drift at a slower rate. Instability is also observed with other frequencies of the disturbance, including a constant $n(t)$.

Rohrs examples stimulated much research about the robustness of adaptive systems. Examination of the mechanisms of instability in Rohrs examples show that the instabilities are related to the identifier. In identification, such instabilities involve computed signals, while in adaptive control, variables associated with the plant are also involved. This justifies a more careful consideration of robustness issues in the context of adaptive control.

5.3 ROBUSTNESS OF ADAPTIVE ALGORITHMS WITH PERSISTENCY OF EXCITATION

Rohrs examples show that the bounded-input bounded-state (BIBS) stability property obtained in Chapter 3 is not robust to uncertainties. In

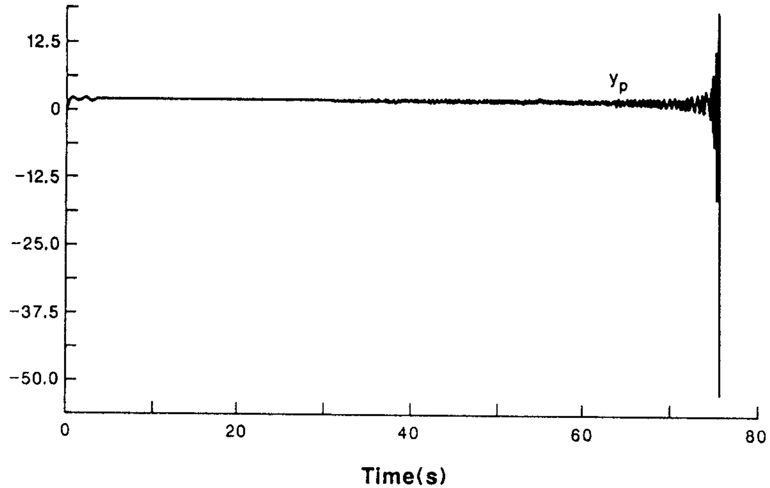

Figure 5.11 Plant Output ($r = 2$, $n = 0.5 \sin 16.1t$)

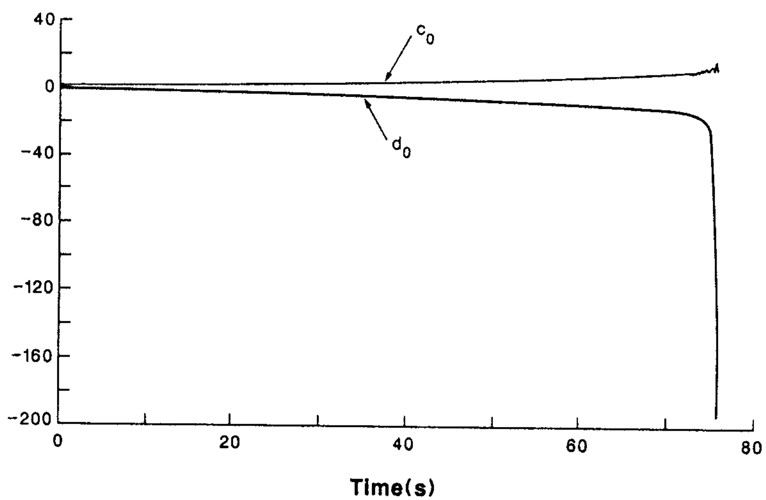

Figure 5.12 Controller Parameters ($r = 2$, $n = 0.5 \sin 16.1t$)

some cases, an arbitrary small disturbance can destabilize an adaptive system, which is otherwise proved to be BIBS stable. In this section, we will show that the property of *exponential stability* is robust, in the sense that exponentially stable systems can tolerate a certain amount of disturbance. Thus, provided that the nominal adaptive system is exponentially stable (guaranteed by a persistency of excitation (PE) condition), we will obtain robustness margins, that is, bounds on disturbances and unmodeled dynamics that do not destroy the stability of the adaptive system. Our presentation follows the lines of Bodson & Sastry [1984].

Of course, the practical notion of robustness is that stability should be preserved in the presence of actual disturbances present in the system. Robustness margins must include actual disturbances for the adaptive system to be robust in that sense. The main difference from classical linear time-invariant (LTI) control system robustness margins is that *robustness does not depend only on the plant and control system, but also on the reference input,* which must guarantee persistent excitation of the nominal adaptive system (that is, without disturbances or unmodeled dynamics).

5.3.1 Exponential Convergence and Robustness

In this section, we consider properties of a so-called *perturbed* system

$$\dot{x} = f(t, x, u) \qquad x(0) = x_0 \tag{5.3.1}$$

and relate its properties to those of the *unperturbed* system

$$\dot{x} = f(t, x, 0) \qquad x(0) = x_0 \tag{5.3.2}$$

where $t \geq 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$. Depending on the interpretation, the signal u will be considered either a disturbance or an input.

We restrict our attention to solutions x and inputs u belonging to some arbitrary balls $B_h \in \mathbb{R}^n$ and $B_c \in \mathbb{R}^m$.

Theorem 5.3.1 Small Signal I/O Stability

Consider the perturbed system (5.3.1) and the unperturbed system (5.3.2). Let $x = 0$ be an equilibrium point of (5.3.2), i.e., $f(t, 0, 0) = 0$, for all $t \geq 0$. Let f be piecewise continuous in t and have continuous and bounded first partial derivatives in x, for all $t \geq 0$, $x \in B_h$, $u \in B_c$. Let f be Lipschitz in u, with Lipschitz constant l_u, for all $t \geq 0$, $x \in B_h$, $u \in B_c$. Let $u \in L_\infty$.

If $x = 0$ is an exponentially stable equilibrium point of the unperturbed system

Then

(a) The perturbed system is *small-signal* L_∞ *- stable,* that is, there exist γ_∞, $c_\infty > 0$, such that $\| u \|_\infty < c_\infty$ implies that

$$\| x \|_\infty \leq \gamma_\infty \| u \|_\infty < h \tag{5.3.3}$$

where x is the solution of (5.3.1) starting at $x_0 = 0$;

(b) There exists $m \geq 1$ such that, for all $|x_0| < h/m$, $0 < \| u \|_\infty < c_\infty$ implies that $x(t)$ converges to a B_δ ball of radius $\delta = \gamma_\infty \| u \|_\infty < h$, that is: for all $\epsilon > 0$, there exists $T \geq 0$ such that

$$|x(t)| \leq (1 + \epsilon)\delta \tag{5.3.4}$$

for all $t \geq T$, along the solutions of (5.3.1) starting at x_0. Also, for all $t \geq 0$, $|x(t)| < h$.

Comments

Part (a) of theorem 5.3.1 is a direct extension of theorem 1 of Vidyasagar & Vannelli [1982] (see also Hill & Moylan [1980]) to the non autonomous case. Part (b) further extends it to non zero initial conditions.

Theorem 5.3.1 relates *internal* exponential stability to *external* input/output stability (the output is here identified with the state). In contrast with the definition of BIBS stability of Section 3.4, we require a linear relationship between the norms in (5.3.3) for L_∞ stability.

Although lack of exponential stability does not imply input/output instability, it is known that simple stability and even (non uniform) asymptotic stability are *not* sufficient conditions to guarantee I/O stability (see e.g., Kalman & Bertram [1960], Ex. 5, p. 379).

Proof of Theorem 5.3.1

The differential equation (5.3.2) satisfies the conditions of theorem 1.5.1, so that there exists a Lyapunov function $v(t, x)$ satisfying the following inequalities

$$\alpha_1 |x|^2 \leq v(t,x) \leq \alpha_2 |x|^2 \tag{5.3.5}$$

$$\left. \frac{dv(t,x)}{dt} \right|_{(5.3.2)} \leq -\alpha_3 |x|^2 \tag{5.3.6}$$

$$\left| \frac{\partial v(t,x)}{\partial x} \right| \leq \alpha_4 |x| \tag{5.3.7}$$

for some strictly positive constants $\alpha_1 \cdots \alpha_4$, and for all $t \geq 0$, $x \in B_h$.

If we consider the same function to study the perturbed differential equation (5.3.1), inequalities (5.3.5) and (5.3.7) still hold, while (5.3.6) is modified, since the derivative is now to be taken along the trajectories of (5.3.1) instead of (5.3.2). The two derivatives are related through

$$\begin{aligned} \left. \frac{dv(t,x)}{dt} \right|_{(5.3.1)} &= \frac{\partial v(t,x)}{\partial t} + \sum_{i=1}^{n} \frac{\partial v(t,x)}{\partial x_i} f_i(t,x,u) \\ &= \left. \frac{dv(t,x)}{dt} \right|_{(5.3.2)} + \sum_{i=1}^{n} \frac{\partial v(t,x)}{\partial x_i} \\ &\quad \cdot \left[f_i(t,x,u) - f_i(t,x,0) \right] \end{aligned} \tag{5.3.8}$$

Using (5.3.5)–(5.3.7), and the Lipschitz condition on f

$$\left. \frac{dv(t,x)}{dt} \right|_{(5.3.1)} \leq -\alpha_3 |x|^2 + \alpha_4 |x| \, l_u \| u \|_\infty \tag{5.3.9}$$

Define

$$\gamma_\infty := \frac{\alpha_4}{\alpha_3} l_u \left[\frac{\alpha_2}{\alpha_1} \right]^{\frac{1}{2}} \tag{5.3.10}$$

$$\delta := \gamma_\infty \| u \|_\infty \tag{5.3.11}$$

$$m := \left[\frac{\alpha_2}{\alpha_1} \right]^{\frac{1}{2}} \geq 1 \tag{5.3.12}$$

Inequality (5.3.9) can now be written

$$\left. \frac{dv(t,x)}{dt} \right|_{(5.3.1)} \leq -\alpha_3 |x| \left[|x| - \frac{\delta}{m} \right] \tag{5.3.13}$$

This inequality is the basis of the proof.

Part (a) Consider the situation when $|x_0| \leq \delta/m$ (this is true in particular if $x_0 = 0$). We show that this implies that $x(t) \in B_\delta$ for all $t \geq 0$ (note that $\delta/m \leq \delta$, since $m \geq 1$).

Suppose, for the sake of contradiction, that it were not true. Then, by continuity of the solutions, there would exist T_0, T_1 ($T_1 > T_0 \geq 0$), such that

$$|x(T_0)| = \delta/m \quad \text{and} \quad |x(T_1)| > \delta$$

and for all $t \in [T_0, T_1] : |x(t)| \geq \delta/m$. Consequently, inequality

(5.3.13) shows that, in $[T_0, T_1]$, $\dot{v} \le 0$. However, this contradicts the fact that

$$v(T_0, x(T_0)) \le \alpha_2(\delta/m)^2 = \alpha_1\delta^2$$

and

$$v(T_1, x(T_1)) > \alpha_1\delta^2$$

Part (b) Assume now that $|x_0| > \delta/m$. We show the result in two steps.
(b1) for all $\epsilon > 0$, there exists $T \ge 0$ such that $|x(T)| \le (\delta/m)(1+\epsilon)$. Suppose it was not true. Then, for some $\epsilon > 0$ and for all $t \ge 0$

$$|x(t)| > (\delta/m)(1+\epsilon)$$

and, from (5.3.13)

$$\dot{v} < -\alpha_3(\delta/m)^2(1+\epsilon)\epsilon$$

which is a strictly negative constant. However, this contradicts the fact that

$$v(0, x_0) \le \alpha_2|x_0|^2 < \alpha_2 \frac{h^2}{m^2}$$

and $v(t, x(t)) \ge 0$ for all $t \ge 0$. Note that an upper bound on T is

$$T \le \frac{\alpha_2 h^2}{\alpha_3\delta^2\epsilon}$$

(b2) for all $t \ge T$, $|x(t)| \le \delta(1+\epsilon)$. This follows directly from (b1), using an argument identical to the one used to prove (a).

Finally, recall that the assumptions require that $x(t) \in B_h$, $u(t) \in B_c$, for all $t \ge 0$. This is also guaranteed, using an argument similar to (a), provided that $|x_0| < h/m$ and $\|u\|_\infty < c_\infty$, where m is defined in (5.3.12), and

$$c_\infty := \min(c, h/\gamma_\infty) \tag{5.3.14}$$

(5.3.14) implies that $\delta < h$, and $|x_0| < h/m \le h$ implies that $|x(t)| \le m|x_0| < h$ for all $t \ge 0$.

Note that although part (a) of the proof is, in itself, a result for non zero initial conditions, the size of the ball $B_{\delta/m}$ involved decreases when the amplitude of the input decreases, while the size of $B_{h/m}$ is independent of it. □

Additional Comments

a) The proof of the theorem gives an interesting interpretation of the interaction between the exponential convergence of the original system and the effect of the disturbances on the perturbed system. To see this, consider (5.3.9): the term $-\alpha_3|x|^2$ acts like a restoring force bringing the state vector back to the origin. This term originates from the exponential stability of the unperturbed system. The term $\alpha_4|x|\, l_u\|u\|_\infty$ acts like a disturbing force, pulling the state *away* from the origin. This term is caused by the input u (i.e. by the disturbance acting on the system). While the first term is proportional to the norm squared, the second is only proportional to the norm, so that when $|x|$ is sufficiently large, the restoring force equilibrates the disturbing force. In the form (5.3.13), we see that this happens when $|x| = \delta/m = \gamma_\infty/m\|u\|_\infty$.

b) If the assumptions are valid *globally*, then the results are valid globally too. The system remains stable and has finite I/O gain, independent of the size of the input. In the example of Section 5.3.2, and for a wide category of nonlinear systems (bilinear systems for example), the Lipschitz condition is not verified globally. Yet, given *any* balls B_h, B_c, the system satisfies a Lipschitz condition with constant l_u depending on the size of the balls (actually increasing with it). The balls B_h, B_c are consequently arbitrary in that case, but the values of γ_∞ (the L_∞ gain) and c_∞ (the stability margin) will vary with them. In general, it can be expected that c_∞ will remain bounded despite the freedom left in the choice of h and c, so that the I/O stability will only be local.

c) *Explicit* values of γ_∞ and c_∞ can be obtained from parameters of the differential equation, using equations (5.3.10) and (5.3.14). Note that if we used the Lyapunov function satisfying (5.3.5)–(5.3.7) to obtain a convergence rate for the unperturbed system, this rate would be $\alpha_3/2\alpha_1$. Therefore, it can be verified that, with other parameters remaining identical, the L_∞ gain is decreased and the stability margin c_∞ is increased, *when the rate of exponential convergence is increased.*

5.3.2 Robustness of an Adaptive Control Scheme

For the purpose of illustration, we consider the output error direct adaptive control algorithm of Section 3.3.2, when the relative degree of the plant is 1. This example contains the specific cases of the Rohrs examples.

In Section 3.5, we showed that the overall output error adaptive scheme for the relative degree 1 case is described by (cf. (3.5.28))

$$\dot{e}(t) = A_m e(t) + b_m \phi^T(t) w_m(t) + b_m \phi^T(t) Q\, e(t)$$

$$\dot{\phi}(t) = -g c_m^T e(t) w_m(t) - g c_m^T e(t) Q\, e(t) \qquad (5.3.15)$$

where $e(t) \in \mathbb{R}^{3n-2}$, and $\phi(t) \in \mathbb{R}^{2n}$. A_m is a stable matrix, and $w_m(t) \in \mathbb{R}^{2n}$ is bounded for all $t \geq 0$. (5.3.15) is a nonlinear ordinary differential equation (actually it is bilinear) of the form

$$\dot{x} = f(t, x) \qquad x(0) = x_0 \qquad (5.3.16)$$

which is of the form (5.3.2), where

$$x := \begin{bmatrix} e \\ \phi \end{bmatrix} \in \mathbb{R}^{5n-2} \qquad (5.3.17)$$

Recall that we also found, in Section 3.8, that (5.3.15) (i.e. (5.3.16)) is exponentially stable in any closed ball, provided that w_m is PE.

Robustness to Output Disturbances

Consider the case when the measured output is affected by a measurement noise $n(t)$, as in Figure 5.6. Denote by y_p^* the output of the plant $\hat{P}_{\theta^*}(s)$ (that is the output without measurement noise) and by $y_p(t)$, the measured output, affected by noise, so that

$$y_p(t) = y_p^*(t) + n(t) = \hat{P}_{\theta^*}(u) + n(t) \qquad (5.3.18)$$

To find a description of the adaptive system in the presence of the measurement noise $n(t)$, we return to the derivation of (5.3.15) (that is (3.5.28)) in Section 3.5. The plant $\hat{P}_{\theta^*}$ has a minimal state-space representation $[A_p, b_p, c_p^T]$ such that

$$\dot{x}_p = A_p x_p + b_p u$$

$$y_p^* = c_p^T x_p \qquad (5.3.19)$$

The observers are described by

$$\dot{w}^{(1)} = \Lambda w^{(1)} + b_\lambda u$$

$$\dot{w}^{(2)} = \Lambda w^{(2)} + b_\lambda y_p$$

$$= \Lambda w^{(2)} + b_\lambda c_p^T x_p + b_\lambda n \qquad (5.3.20)$$

and the control input is given by $u = \theta^T w = \phi^T w + \theta^{*T} w$.

As previously, we let $x_{pw}^T = (x_p^T, w^{(1)^T}, w^{(2)^T})$. Using the definition of A_m, b_m and c_m in (3.5.18)–(3.5.19), the description of the plant with controller is now

$$\dot{x}_{pw} = A_m x_{pw} + b_m \phi^T w + b_m c_0^* r + b_n n$$

$$y_p^* = c_m^T x_{pw} \qquad (5.3.21)$$

where we define $b_n^T = (0, 0, b_\lambda^T) \in (\mathbb{R}^n, \mathbb{R}^{n-1}, \mathbb{R}^{n-1}) = \mathbb{R}^{3n-2}$.

As previously, we represent the model and its output by

$$\dot{x}_m = A_m x_m + b_m c_0^* r$$

$$y_m = c_m^T x_m \qquad (5.3.22)$$

and we let $e = x_{pw} - x_m$.

The update law is given by

$$\dot{\phi} = -g(y_p - y_m) w$$

$$= -g c_m^T e w - g n w \qquad (5.3.23)$$

and the regressor is now related to the state e by

$$w = \begin{bmatrix} r \\ w^{(1)} \\ y_p \\ w^{(2)} \end{bmatrix} = w_m + \begin{bmatrix} 0 \\ w^{(1)} - w_m^{(1)} \\ y_p^* - y_m \\ w^{(2)} - w_m^{(2)} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ n \\ 0 \end{bmatrix}$$

$$= w_m + Qe + q_n n \qquad (5.3.24)$$

where we define $q_n^T = (0, 0, 1, 0) \in (\mathbb{R}, \mathbb{R}^{n-1}, \mathbb{R}, \mathbb{R}^{n+1}) = \mathbb{R}^{2n}$.

Using these results, the adaptive system with measurement noise is described by

$$\dot{e}(t) = A_m e(t) + b_m \phi^T(t) w_m(t) + b_m \phi^T(t) Q e(t) + b_m \phi^T(t) q_n n(t) + b_n n(t)$$

$$\dot{\phi}(t) = -g c_m^T e(t) w_m(t) - g c_m^T e(t) Q e(t) - g c_m^T e(t) q_n n(t) - g n(t) w_m(t) - g n(t) Q e(t) - g n^2(t) q_n \qquad (5.3.25)$$

which, with the definition of x in (5.3.17) and the definition of f in (5.3.15)–(5.3.16) can be written

$$\dot{x} = f(t,x) + p_1(t) + P_2(t)x(t) \tag{5.3.26}$$

where $p_1(t) \in \mathbb{R}^{5n-2}$ and $P_2(t) \in \mathbb{R}^{5n-2 \times 5n-2}$ are given by

$$p_1(t) = \begin{bmatrix} b_n n(t) \\ -gn(t)w_m(t) - gn^2(t)q_n \end{bmatrix}$$

$$P_2(t) = \begin{bmatrix} 0 & b_m n(t) q_n^T \\ -gn(t)q_n c_m^T - gn(t)Q & 0 \end{bmatrix} \tag{5.3.27}$$

Note that if $n \in L_\infty$, then p_1 and $P_2 \in L_\infty$. Therefore, the perturbed system (5.3.26) is a special form of system (5.3.1), where u contains the components of p_1 and P_2. Although $p_1(t)$ depends quadratically on n, given a bound on n, there exists $k_n \geq 0$ such that

$$\| p_1 \|_\infty + \| P_2 \|_\infty \leq k_n \| n \|_\infty \tag{5.3.28}$$

From these derivations, we deduce the following theorem.

Theorem 5.3.2 Robustness to Disturbances

Consider the output error direct adaptive control scheme of Section 3.2.2, assuming that the relative degree of the plant is 1. Assume that the measured output y_p of the plant is given by (5.3.18), where $n \in L_\infty$. Let $h > 0$.

If w_m is PE

Then there exists $\gamma_n, c_n > 0$ and $m \geq 1$, such that $\| n \|_\infty < c_n$ and $|x(0)| < h/m$ implies that $x(t)$ converges to a B_δ ball of radius $\delta = \gamma_n \| n \|_\infty$, with $|x(t)| \leq m|x_0| < h$ for all $t \geq 0$.

Proof of Theorem 5.3.2

Since w_m is PE, the unperturbed system (5.3.15) (i.e. (5.3.16)) is exponentially stable in any B_h by theorem 3.8.2. The perturbed system (5.3.25) (i.e. (5.3.26)) is a special case of the general form (5.3.1), so that theorem 5.3.1 can be applied with u containing the components of $p_1(t), P_2(t)$. The results on $p_1(t), P_2(t)$ can be translated into similar results involving $n(t)$, using (5.3.28). □

Comments

a) A specific bound c_n on $\| n \|_\infty$ can be obtained such that, within this bound, and provided the initial error is sufficiently small, *the stability of the adaptive system will be preserved.* For this reason, c_n is called a robustness margin of the adaptive system to output disturbances.

b) The deviations from equilibrium are locally *at most proportional* to the disturbances (in terms of L_∞ norms), and their bounds can be made arbitrarily small by reducing the bounds on the disturbances.

c) The L_∞ gain from the disturbances to the deviations from equilibrium can be reduced by *increasing the rate of exponential convergence of the unperturbed system* (provided that other constants remain identical).

d) Rohrs example (R3) of instability of an adaptive scheme with output disturbances on a non persistently excited system, is an example of instability when the persistency of excitation condition of the nominal system is not satisfied.

Robustness to Unmodeled Dynamics

We assume again that there exists a nominal plant $\hat{P}_{\theta^*}(s)$, satisfying the assumptions on which the adaptive control scheme is based, and we define the *output of the nominal plant* to be

$$y_p^* = \hat{P}_{\theta^*}(u) \tag{5.3.29}$$

The actual output is modeled as the output of the nominal plant, plus some additive uncertainty represented by a bounded operator H_a

$$y_p(t) = y_p^*(t) + H_a(u)(t) \tag{5.3.30}$$

The operator H_a represents the difference between the real plant, and the idealized plant $\hat{P}(s)$. We refer to it as an *additive unstructured uncertainty,* and it constitutes all the uncertainty, since it is the purpose of the adaptive scheme to reduce to zero the *structured* or *parametric* uncertainty.

We assume that $H_a : L_{\infty e} \to L_{\infty e}$ is a causal operator satisfying

$$\| H_a(u)_t \|_\infty \leq \gamma_a \| u_t \|_\infty + \beta_a \tag{5.3.31}$$

for all $t \geq 0$. β_a may include the effect of initial conditions in the unmodeled dynamics and the possible presence of bounded output disturbances.

The following theorem guarantees the stability of the adaptive system in the presence of unmodeled dynamics satisfying (5.3.31).

Theorem 5.3.3 Robustness to Unmodeled Dynamics
Consider the output error direct adaptive control scheme of Section 3.3.2, assuming that the relative degree of the plant is 1. Assume that the nominal plant output and actual measured plant output satisfy (5.3.29)–(5.3.30), where $\hat{P}_{\theta^*}$ satisfies the assumptions of Section 3.3.2. H_a satisfies (5.3.31) and is such that trajectories of the adaptive system are continuous with respect to t.

If w_m is PE

Then for x_0, γ_a, β_a sufficiently small, the state trajectories of the adaptive system remain bounded.

Proof of Theorem 5.3.3
Let $T > 0$ such that $x(t) \le h$ for all $t \in [0, T]$. Define $n = H_a(u)$, so that, by assumption

$$\| n_t \|_\infty \le \gamma_a \| u_t \|_\infty + \beta_a \qquad (5.3.32)$$

for all $t \in [0, T]$. Using (5.3.24), the input u is given by

$$u = \theta^T w = \theta^{*T} w + \phi^T w$$
$$= \theta^{*T} w_m + \theta^{*T} Qe + \theta^{*T} q_n n + \phi^T w_m + \phi^T Qe + \ + \phi^T q_n n \qquad (5.3.33)$$

Since $x \in B_h$, there exist $\gamma_u, \beta_u \ge 0$ such that

$$\| u_t \|_\infty \le \gamma_u \| n_t \|_\infty + \beta_u \qquad (5.3.34)$$

for all $t \in [0, T]$. Let γ_a, β_a sufficiently small that

$$\gamma_a \gamma_u < 1 \qquad (5.3.35)$$

$$\frac{\beta_a + \gamma_a \beta_u}{1 - \gamma_a \gamma_u} < c_n \qquad (5.3.36)$$

where c_n is the constant found in theorem 5.3.2. Applying the small gain theorem (lemma 3.6.6), and using (5.3.32), (5.3.35) and (5.3.36), it follows that $\| n_t \|_\infty < c_n$. By theorem 5.3.2, this implies that $|x(t)| < h$ for all $t \in [0, T]$. Since none of the constants γ_a, β_a, γ_n and β_n is dependent on T, $|x(t)| < h$ for all $t \ge 0$. Indeed, suppose it was not true. Then, by continuity of the solutions, there would exist a $T > 0$ such that $|x(t)| \le h$ for all $t \in [0, T]$, and $x(T) = h$. The theorem would then apply, resulting in a contradiction since $|x(T)| < h$. □

Comments
Condition (5.3.24) is very general, since it includes possible nonlinearities, unmodeled dynamics, and so on, provided that they can be represented by additive, bounded-input bounded-output operators.

If the operator H_a is linear time invariant, the stability condition is a condition on the L_∞ gain of H_a. One can use

$$\gamma_a = \| h_a \|_1 = \int_0^\infty | h_a(\tau) | \, d\tau \qquad (5.3.37)$$

where $h_a(\tau)$ is the impulse response of $\hat{H}_a$. The constant β_a depends on the initial conditions in the unmodeled dynamics.

The proof of theorem 5.3.3 gives some margins of unmodeled dynamics that can be tolerated without loss of stability of the adaptive system. Given γ_a, β_a it is actually possible to compute these values. The most difficult parameter to determine is possibly the rate of convergence of the unperturbed system, but we saw in Chapter 4 how some estimate could be obtained, under the conditions of averaging. Needless to say the expression for these robustness margins depends in a complex way on unknown parameters, and it is likely that the estimates would be conservative. The importance of the result is to show that if the unperturbed system is persistently excited, it will tolerate *some* amount of disturbance, or conversely that an arbitrary small disturbance *cannot* destabilize the system, such as in example (R3).

5.4 HEURISTIC ANALYSIS OF THE ROHRS EXAMPLES

By considering the overall adaptive system, including the plant states, observer states, and the adaptive parameters, we showed in Section 5.3 the importance of the exponential convergence to guarantee some robustness of the adaptive system. This convergence depends especially on the *parameter* convergence, and therefore on conditions on the input signal $r(t)$.

A heuristic analysis of the Rohrs examples gives additional insight into the mechanisms leading to instability, and suggest practical methods to improve robustness. Such an analysis can be found in Astrom [1983], and its success relies mainly on the separation of time scales between the evolution of the plant/observer states, and the evolution of the adaptive parameters. This separation of time scales is especially suited for the application of averaging methods (cf. Chapter 4).

Following Astrom [1983], we will show that instability in the Rohrs examples are due to one or more of the following factors

a) the lack of sufficiently rich inputs to

- allow for parameter convergence in the nominal system,
- prevent the drift of the parameters due to unmodeled dynamics or output disturbances.

b) the presence of significant excitation at high frequencies, originating either from the reference input, or from output disturbances. These signals cause the adaptive loop to try to get the plant loop to match the model at high frequencies, resulting in a closed-loop unstable plant.

c) a large reference input with a non-normalized identification (adaptation) algorithm and unmodeled dynamics, resulting in the instability of the identification algorithm.

Analysis

Consider now the mechanisms of instability corresponding to these three cases.

a) Consider first the case when the input is not sufficiently rich (example (R3)).

In the nominal case, the output error tends to zero. When the PE condition is not satisfied, the controller parameter does not necessarily converge to its nominal value, but to a value such that the closed-loop transfer function matches the model transfer function at the frequencies of the reference input. Consider Rohrs example, without unmodeled dynamics. The closed-loop transfer function from $r \to y_p$, assuming that c_0 and d_0 are fixed, is

$$\frac{\hat{y}_p}{\hat{r}} = \frac{2c_0}{s+1-2d_0} \tag{5.4.1}$$

If a constant reference input is used, only the DC gain of this transfer function must be matched with the DC gain of the reference model. This implies the condition that

$$\frac{2c_0}{1-2d_0} = 1 \tag{5.4.2}$$

Any value of c_0, d_0 satisfying (5.4.2) will lead to $y_p - y_m \to 0$ as $t \to \infty$ for a constant reference input. Conversely, when $e_0 \to 0$, so do $\dot{c}_0$, and $\dot{d}_0$, so that the assumption that c_0, d_0 are fixed is justified.

If an output disturbance $n(t)$ enters the adaptive system, it can cause the parameters c_0, d_0 to move along the line (more generally the surface) defined by (5.4.2), leaving $e_0 = y_p - y_m$ at zero. In particular, note that when output disturbances are present, the actual update law for d_0 is not (5.2.6) anymore, but

$$\dot{d}_0 = -g y_p^*(y_p^* - y_m) - g y_m n - g n^2 \tag{5.4.3}$$

where we find the presence of the term $-gn^2$, which will tend to make d_0 slowly drift toward the negative direction.

In example (R3), unmodeled dynamics are present, so that the transfer function from $r \to y_p$ is in fact given by

$$\frac{\hat{y}_p}{\hat{r}} = \frac{458c_0}{(s+1)(s^2+30s+229)-458d_0} \tag{5.4.4}$$

which is identical to (5.4.1) for DC signals, but which is unstable for $d_0 \geq 0.5$ and $d_0 \leq -17.03$.

The result is observed in Figures 5.11 and 5.12, where d_0 slowly drifts in the negative direction, until it reaches the limit of stability of the closed-loop plant with unmodeled dynamics. This instability is called the *slow drift instability.* The error converges to a neighborhood of zero, and the signal available for parameter update is very small and unreliable, since it is indistinguishable from the output noise $n(t)$. It is the accumulation of updates based on incorrect information that leads to parameter drift, and eventually to instability.

In terms of the discussion of Section 5.3, we see that the constant disturbance $-gn^2$ is not counteracted by any restoring force, as would be the case if the original system was exponentially stable. For example, consider the case where $n = 0.1 \sin 16.1\, t$. Figure 5.13 shows the evolution of the parameter d_0 in a simulation where $r(t) = 2$ and where $r(t) = 2\sin t$. In the first case, the parameter slowly drifts, leading eventually to instability. When $r(t) = 2\sin t$, so that PE conditions are satisfied, the parameter d_0 deviates from d_0^* but remains close to the nominal value.

Finally, note that instabilities of this type can be obtained for systems of relative degree greater than two even without unmodeled dynamics and can lead to the so-called *bursting phenomenon* (cf. Anderson [1985]). The presence of noise in the update law leads to drift in the feedback coefficients to a region where they are large, resulting in a closed loop unstable system and a large increase in e_0. The output error e_0 eventually converges back to zero, but a large 'blip' is observed. This repeats at random instants, and is referred to as bursting. As before a safeguard, against bursting is persistent excitation.

b) Consider now the case when the reference input, or the output disturbance, contain a large component at a frequency where unmodeled dynamics are significant (example (R2)).

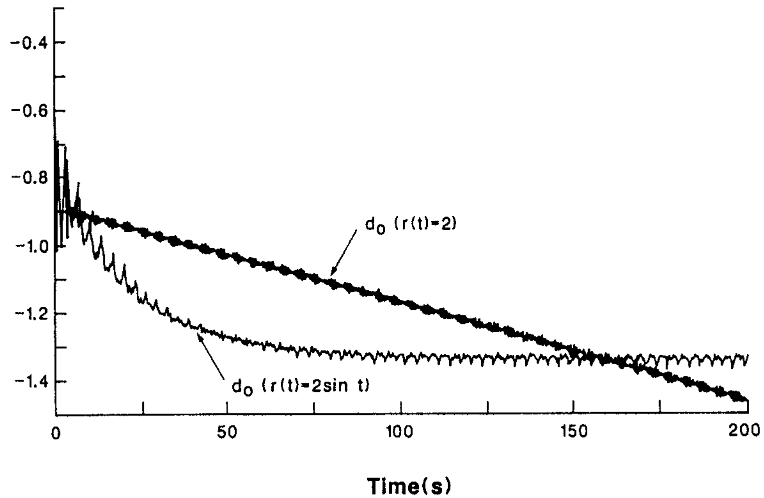

Figure 5.13 Controller Parameter $d_0(n = 0.1 \sin 16.1t)$

Let us return to Rohrs example, with a sinusoidal reference input $r(t) = r_0 \sin(\omega_0 t)$. With unmodeled dynamics, there are still *unique* values of c_0, d_0 such that the transfer function from $r \to y_p$ matches $\hat{M}$ at the frequency of the reference input ω_0. Without unmodeled dynamics, these would be the nominal c_0^*, d_0^*, but now they are the values c_0^+, d_0^+ given by

$$\left. \frac{458 c_0^+}{(s+1)(s^2+30s+229) - 458 d_0^+} \right|_{j\omega_0} = \left. \frac{3}{s+3} \right|_{j\omega_0} \tag{5.4.5}$$

where ω_0 is the frequency of the reference input. Note that the values of c_0^+, d_0^+ depend on $\hat{M}$, $\hat{P}$, the unmodeled dynamics, and also *on the reference input* r.

On the other hand, it may be verified through simulations that the output error tends to zero and that the controller parameters converge to the following values $c_{0_{ss}}$ and $d_{0_{ss}}$ (cf. Astrom [1983]).

ω_0	$c_{0_{ss}}$	$d_{0_{ss}}$
1	1.69	-1.26
2	1.67	-1.44
5	1.53	-2.72
10	1.04	-7.31

It may be verified that these values are identical to c_0^+, d_0^+ defined earlier. Therefore, the adaptive control system updates the parameters, trying to match the closed-loop transfer function—including the unmodeled dynamics—to the model reference transfer function. Note that the parameter $d_{0_{ss}} = d_0^+$ quickly decreases for $\omega_0 > 5$. On the other hand, the closed-loop system is unstable when $d_0 \leq -17.03$ and $d_0^+ \leq -17.03$, when $\omega_0 \geq 16.09$. Therefore, by attempting to match the reference model at a high frequency, the adaptive system leads to an unstable closed-loop system, and thereby to an unstable overall system.

This is the instability observed in example (R2). In contrast, Figure 5.14 shows a simulation where $r = 0.3 + 1.85 \sin t$, that is where the sinusoidal component of the input is at a frequency where model matching is possible.

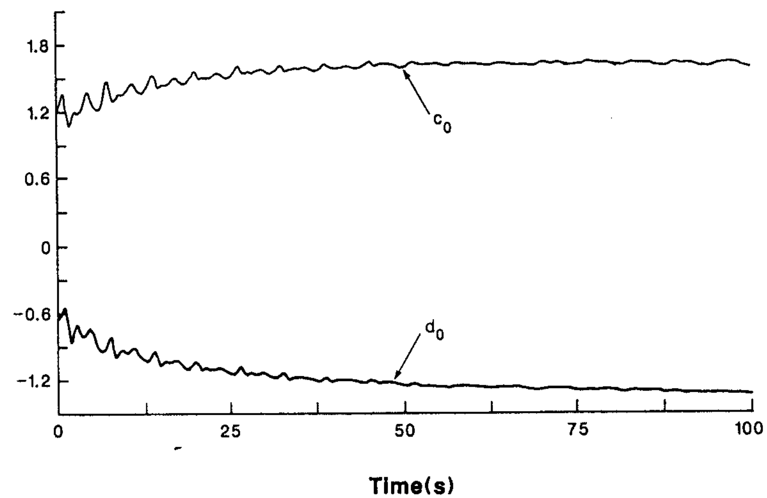

Figure 5.14 Controller Parameters $(r = 0.3 + 1.85 \sin t\,,\ n = 0)$

Then, the parameters converge to values c_0^+, d_0^+ close to c_0^*, d_0^*, and the adaptive system remains stable, despite the unmodeled dynamics.

c) Consider finally the mechanism of instability observed with a large reference input (example (R1)).

This mechanism will be called the *high-gain identifier instability.* Although we do not have explicitly a high adaptation gain g, recall that the adaptation law is given by

$$\dot{c}_0 = -g e_0 r \tag{5.4.6}$$

$$\dot{d}_0 = -g e_0 y_p \tag{5.4.7}$$

Roughly speaking, multiplying r by 2 means multiplying y_m, y_p and e_0 by 2 and therefore is equivalent to multiplying the adaptation gain by 4.

The instabilities obtained for high values of the adaptation gain are comparable to instabilities caused by high gain feedback in LTI systems with relative degree greater than 2 (cf. Astrom [1983] for a simple root-locus argument). A simple fix to these problems is to replace the identification algorithm by a *normalized* algorithm.

5.5 AVERAGING ANALYSIS OF SLOW DRIFT INSTABILITY

As was pointed out in Section 5.4, Astrom [1983] introduced an analysis of instability based on slow parameter adaptation, to separate the evolution of the plant/observer states and the adaptive parameters. The phenomenon under study is the so-called *slow drift instability* and is caused by either a lack of sufficiently rich inputs, or the presence of significant excitation at high frequencies, originating either from the reference input or output disturbances.

A heuristic analysis of this phenomenon was already given in the preceding section. In this section, we make the analysis more rigorous using the averaging framework of Chapter 4. In Section 5.5.1, we develop general instability theorems for averaging of one and two time scale systems. In Section 5.5.2, we apply these results to an output error adaptive scheme. Our treatment is based on Riedle & Kokotovic [1985] and Fu & Sastry [1986].

5.5.1 Instability Theorems Using Averaging

One Time Scale Systems

Recall the setup of Section 4.2, where we considered differential equations of the form

$$\dot{x} = \epsilon f(t, x, \epsilon) \tag{5.5.1}$$

and their *averaged* versions

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}) \tag{5.5.2}$$

where

$$f_{av}(x) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f(\tau, x, 0)\, d\tau \tag{5.5.3}$$

assuming that the limit exists uniformly in t_0 and x. We will not repeat the definitions and the assumptions (A1)–(A5) of Section 4.2, but we will assume that the systems (5.5.1), (5.5.2) satisfy those identical assumptions. The reader may wish to review those assumptions before proceeding with the proof of the following theorem.

Theorem 5.5.1 Instability Theorem for One Time Scale Systems

If the original system (5.5.1) and the averaged system (5.5.2) satisfy assumptions (A1)–(A5), the function $f_{av}(x)$ has continuous and bounded first partial derivatives in x, and there exists a continuously differentiable, decrescent function $v(t, x)$ such that

(i) $v(t, 0) = 0$

(ii) $v(t, x) > 0$ for some x arbitrarily close to 0

(iii) $\left| \frac{\partial v(t, x)}{\partial x} \right| \le k_1 |x|$ for some $k_1 > 0$

(iv) the derivative of $v(t, x)$ along the trajectories of (5.5.2) satisfies

$$\dot{v}(t, x)\Big|_{(5.5.2)} \ge \epsilon k_2 |x|^2 \tag{5.5.4}$$

for some $k_2 > 0$.

Then the original system (5.5.1) is unstable for ϵ sufficiently small.

Remark

By an instability theorem of Lyapunov (see for example Vidyasagar [1978]), the additional assumptions (i)–(iv) of the theorem guarantee that the averaged system (5.5.2) is unstable. By definition, a system is *unstable* if it is not stable, meaning that there exists a neighborhood of the origin and arbitrarily small initial conditions so that the state vectors originating from them are expelled from the neighborhood of the origin.

Proof of Theorem 5.5.1

As in Chapter 4, the first step is to use the transformation of lemma 4.2.3 to transform the original system. Thus, we use

$$x = z + \epsilon w_\epsilon(t, z) \tag{5.5.5}$$

with $w_\epsilon(t, z)$ satisfying, for some $\xi(\epsilon) \in K$

$$|\epsilon w_\epsilon(t,z)| \le \xi(\epsilon)|z| \tag{5.5.6}$$

$$\left|\epsilon \frac{\partial w_\epsilon(t,z)}{\partial z}\right| \le \xi(\epsilon) \tag{5.5.7}$$

to transform (5.5.1) into

$$\dot{z} = \epsilon f_{av}(z) + \epsilon p(t,z,\epsilon) \tag{5.5.8}$$

where $p(t, z, \epsilon)$ satisfies

$$|p(t,z,\epsilon)| \le \psi(\epsilon)|z| \tag{5.5.9}$$

for some $\psi(\epsilon) \in K$.

Now, consider the derivative of $v(t, z)$ along the trajectories of (5.5.8), namely

$$\dot{v}(t,z)\Big|_{(5.5.8)} = \dot{v}(t,z)\Big|_{(5.5.2)} + \frac{\partial v(t,z)}{\partial z}\,\epsilon p(t,z,\epsilon) \tag{5.5.10}$$

Using the inequalities (5.5.4) and (5.5.9), we have that

$$\dot{v}(t,z)\Big|_{(5.5.8)} \ge \epsilon k_2|z|^2 - \epsilon\psi(\epsilon)k_1|z|^2 \tag{5.5.11}$$

If ϵ_0 is chosen so that $k_2 - \psi(\epsilon_0)k_1 > 0$, then it is clear that $\dot{v}(t,z)|_{(5.5.8)}$ is positive definite. By the same Lyapunov instability theorem as was mentioned in the remark preceding the theorem, it follows from (5.5.11) that for $\epsilon \le \epsilon_0$, the system (5.5.8) is unstable, and consequently so is the original system (5.5.1). □

Comments

The continuously differentiable, decrescent function required by theorem 5.5.1 can be found prescriptively, if the averaged system is linear, that is

$$\dot{x}_{av} = \epsilon A x_{av} \tag{5.5.12}$$

and if A has at least one eigenvalue in the open right-half plane, but no eigenvalue on the $j\omega$-axis. In this case, the function v can be chosen to be

$$v(x) = x^T P\, x$$

where P satisfies the Lyapunov equation

$$A^T P + PA = I \tag{5.5.13}$$

The *Taussky lemma generalized* (see Vidyasagar [1978]) says that P has at least one positive eigenvalue, so that $v(x)$ takes on positive values in some directions (and arbitrarily close to the origin). It is also easy to verify that the conditions (iii), (iv) of theorem 5.5.1 are also satisfied by $v(x)$.

Two Time Scale Systems

We now consider the system of Section 4.4 namely

$$\dot{x} = \epsilon f(t,x,y) \tag{5.5.14}$$

$$\dot{y} = A\,y + \epsilon g(t,x,y) \tag{5.5.15}$$

with $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$. The only difference between (5.5.14), (5.5.15) and the system (4.4.1), (4.4.2) of Chapter 4 is that the A matrix of (5.5.15) is now assumed to be constant and stable rather than a function of x which is uniformly stable. The averaged system is

$$\dot{x}_{av} = \epsilon f_{av}(x_{av}) \tag{5.5.16}$$

where $f_{av}(x)$ is defined to be

$$f_{av}(x) = \lim_{T\to\infty} \frac{1}{T}\int_{t_0}^{t_0+T} f(\tau,x,0)\,d\tau \tag{5.5.17}$$

The functions f, g satisfy assumptions (B1), (B2), (B3) and (B5) (only assumption (B4) is not necessary). As in the case of theorem 5.5.1, we advise the reader to review the results of Section 4.4 before following the next theorem.

Theorem 5.5.2 Instability Theorem for Two Time Scale Systems

If the original system (5.5.14), (5.5.15) and the averaged system (5.5.16) satisfy assumptions (B1), (B2), (B3) and (B5), along with the assumption that there exists a continuously differentiable decrescent function $v(t, x)$ such that

(i) $v(t, 0) = 0$

(ii) $v(t, x) > 0$ for some x arbitrarily close to 0

(iii) $\left|\dfrac{\partial v(t,x)}{\partial x}\right| \le k_1|x|$ for some $k_1 > 0$

(iv) the derivative of $v(t, x)$ along the trajectory of (5.5.16) satisfies

$$\dot{v}(t,x)\Big|_{(5.5.16)} \ge \epsilon k_2|x|^2 \tag{5.5.18}$$

for some $k_2 > 0$.

Then the original system (5.5.14), (5.5.15) is unstable for ϵ sufficiently small.

Proof of Theorem 5.5.2

To study the instability of (5.5.14), (5.5.15), we consider another decrescent function v_1,

$$v_1(t, x, y) = v(t, x) - k_3 y^T P y \tag{5.5.19}$$

where P is the symmetric positive definite matrix satisfying the Lyapunov equation

$$A^T P + P A = -I$$

Using the transformation of lemma 4.4.1, we may transform (5.5.14), (5.5.15)—as in the proof of theorems 4.4.2 and 4.4.3 —into

$$\dot z = \epsilon f_{av}(z) + \epsilon p_1(t, z, \epsilon) + \epsilon p_2(t, z, y, \epsilon) \tag{5.5.20}$$

$$\dot y = Ay + \epsilon g(t, x(z), y) \tag{5.5.21}$$

where $p_1(t, z, \epsilon)$ and $p_2(t, z, y, \epsilon)$ satisfy

$$|p_1(t, z, \epsilon)| \le \xi(\epsilon) k_4 |z| \tag{5.5.22}$$

$$|p_2(t, z, y, \epsilon)| \le k_5 |y| \tag{5.5.23}$$

and $\xi(\epsilon) \in K$. Clearly, $v_1(t, z, y) > 0$ for some (x, y) values arbitrarily close to the origin (let $y = 0$ and use assumption (ii)). Now, consider

$$\dot v_1(t, z, y)\Big|_{(5.5.20,\ 21)} = \dot v(t, z)\Big|_{(5.5.20)} + k_3 |y|^2 - 2\epsilon k_3 y^T P g(t, z, y)$$

Using exactly the same techniques as in the proof of theorem 4.4.3 (the reader may wish to follow through the details), it may be verified that

$$\dot v_1(t, z, y)\Big|_{(5.5.20,\ 21)} \ge \epsilon \alpha(\epsilon) |z|^2 + q(\epsilon) |y|^2$$

for some $\alpha(\epsilon) \to k_2$ and $q(\epsilon) \to k_3$ as $\epsilon \to 0$. Thus $\dot v_1(t, z, y)$ is a positive definite function along the trajectories of (5.5.20, 21). Hence, the system (5.5.20), (5.5.21) and consequently the original system (5.5.14), (5.5.15) is unstable for ϵ sufficiently small. □

Mixed Time Scales

As was noted in Chapter 4, a more general class of two-time scale systems arises in adaptive control, having the form

$$\dot x = \epsilon f'(t, x, y') \tag{5.5.24}$$

$$\dot y' = Ay' + h(t, x) + \epsilon g'(t, x, y') \tag{5.5.25}$$

Again, for simplicity, we let the matrix A be a constant matrix (we will only consider linearized adaptive control schemes in the next section).

In (5.5.24), (5.5.25), x is the slow variable but y' has both a fast and a slow component. As we saw in Section 4.4, the system (5.5.24), (5.5.25) can be transformed into the system (5.5.20), (5.5.21) through the use of the coordinate change

$$y = y' - v(t, x) \tag{5.5.26}$$

where $v(t, x)$ is defined to be

$$v(t, x) := \int_0^t e^{A(t-\tau)} h(\tau, x)\, d\tau \tag{5.5.27}$$

The averaged system of (5.5.24), (5.5.25) of the form of (5.5.16) will exist if the following limit exists uniformly in t_0 and x

$$f_{av}(x) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f'(\tau, x, v(\tau, x))\, d\tau$$

The instability theorem of 5.5.2 is applicable with the additional assumption (B6) of Section 4.4.

5.5.2 Application to the Output Error Scheme

Tuned Error Formulation with Unmodeled Dynamics

We will apply the results of the previous section to an output error adaptive control scheme (see Section 3.3.2) designed for a plant of order n and relative degree one. The controller is, however, applied to a plant of order $n + n_u$, where the extra n_u states represent the unmodeled dynamics. In analogy to (3.5.16), the total state of the plant and observers $x_{pw} \in \mathbb{R}^{3n-2+n_u}$ satisfies the equations

$$\begin{bmatrix} \dot x_p \\ \dot w^{(1)} \\ \dot w^{(2)} \end{bmatrix} = \begin{bmatrix} A_p & 0 & 0 \\ 0 & \Lambda & 0 \\ b_\lambda c_p^T & 0 & \Lambda \end{bmatrix} \begin{bmatrix} x_p \\ w^{(1)} \\ w^{(2)} \end{bmatrix} + \begin{bmatrix} b_p \\ b_\lambda \\ 0 \end{bmatrix} u$$

$$y_p = [c_p^T\ 0\ 0] \begin{bmatrix} x_p \\ w^{(1)} \\ w^{(2)} \end{bmatrix} \tag{5.5.28}$$

(5.5.28) is precisely (3.5.16) with the difference that $x_p \in \mathbb{R}^{n+n_u}$ rather to $\mathbb{R}^n$.

Now, it may no longer be possible to find a $\theta^* \in \mathbb{R}^{2n}$ such that the closed loop plant transfer function equals the model transfer function. Instead, we will assume that there is a value of θ which is at least *stabilizes* the closed loop system, and refer to it as the *tuned value* θ_*. We define

$$A_* = \begin{bmatrix} A_p + b_p d_{0*} c_p^T & b_\lambda c_*^T & b_p d_*^T \\ b_\lambda d_{0*} c_p^T & \Lambda + b_\lambda c_*^T & b_\lambda d_*^T \\ b_\lambda c_p^T & 0 & \Lambda \end{bmatrix} \tag{5.5.29}$$

We will call A_* the *tuned closed-loop matrix* (cf. (3.5.18)), and we define the *tuned plant* as

$$\begin{aligned} \dot{x}_{pw*} &= A_* x_{pw*} + b_* c_{0*} r \\ y_{p*} &= c_*^T x_{pw*} \end{aligned} \tag{5.5.30}$$

where

$$b_* = \begin{bmatrix} b_p \\ b_\lambda \\ 0 \end{bmatrix} \in \mathbb{R}^{3n+n_u-2} \quad \text{and} \quad c_* = \begin{bmatrix} c_p \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^{3n+n_u-2}$$

Note the analogy between (5.5.30) and (3.5.20). Now, the transfer function of the tuned plant is not exactly equal to the transfer function of the model, and the error between the tuned plant output and the model output is referred to as the *tuned error*

$$e_* = y_{p*} - y_m \tag{5.5.31}$$

Typically, the values θ_* which are chosen as tuned values correspond to those values of θ_* for which the tuned plant transfer function approximately matches the model transfer function at low frequencies (at those frequencies, the effect of unmodeled dynamics is small).

An error formulation may now be derived, with respect to the tuned system instead of the model system of Section 3.5. Let $\tilde{\theta} := \theta - \theta_*$ represent the parameter error with respect to the tuned parameter value, and rewrite (5.5.28) as

$$\begin{aligned} \dot{x}_{pw} &= A_* x_{pw} + b_* \tilde{\theta}^T w + b_* c_{0*} r \\ y_p &= c_*^T x_{pw} \end{aligned} \tag{5.5.32}$$

This is similar to (3.5.19). The output error parameter update law is

$$\dot{\theta} = \dot{\tilde{\theta}} = -g e_0 w \tag{5.5.33}$$

In turn the output error $e_0 := y_p - y_m$ can be decomposed as

$$e_0 = y_p - y_m = y_p - y_{p*} + e_* \tag{5.5.34}$$

Now, defining $\tilde{e} = x_{pw} - x_{pw*}$ and $\tilde{e}_0 = y_p - y_{p*}$ we may subtract equation (5.5.30) from equation (5.5.31) to get

$$\begin{aligned} \dot{\tilde{e}} &= A_* \tilde{e} + b_* \tilde{\theta}^T w \\ \tilde{e}_0 &= c_*^T \tilde{e} \end{aligned} \tag{5.5.35}$$

along with the update law

$$\dot{\tilde{\theta}} = -g(c_*^T \tilde{e} + e_*) w \tag{5.5.36}$$

As in the ideal case, w can be written as $w = w_* + Q\tilde{e}$ (with $w_* \in \mathbb{R}^{2n}$ having the obvious interpretation), so that (5.5.35), (5.5.36) may be combined to yield

$$\begin{aligned} \dot{\tilde{e}} &= A_* \tilde{e} + b_* \tilde{\theta}^T w_* + b_* \tilde{\theta}^T Q \tilde{e} \\ \dot{\tilde{\theta}} &= -g c_*^T \tilde{e} w_* - g c_*^T \tilde{e} Q \tilde{e} - g e_* w_* - g e_* Q \tilde{e} \end{aligned} \tag{5.5.37}$$

Comparing the equations (5.5.37) with the corresponding equation (3.5.28) for the adaptive system, one sees the presence of two new terms in the second equation. If the tuned error $e_* = 0$, the terms disappear and the equations (5.5.37) reduce to (3.5.28). The first of the new terms is an exogenous forcing term and the second a term which is linear in the error state variables. Without the term $e_* w_*$, the origin $\tilde{e} = 0, \tilde{\theta} = 0$ is an equilibrium of the system. Consequently, we will drop this term for the sake of our local stability/instability analysis. We will also treat the second term $g e_* Q \tilde{e}$ as a small perturbation term (which it is if e_* is small) and focus attention on the linearized and simplified system

$$\begin{aligned} \dot{\tilde{e}} &= A_* \tilde{e} + b_* w_*^T \tilde{\theta} \\ \dot{\tilde{\theta}} &= -g c_*^T \tilde{e} w_* \end{aligned} \tag{5.5.38}$$

Averaging Analysis

To apply the averaging theory of the previous section, we set the gain $g = \epsilon$, a small parameter. Since A_* is stable, it is easy to see that the system (5.5.38) is of the form of the mixed time scale system (5.5.24), (5.5.25) so that averaging may be applied. The averaged parameter error

$\tilde{\theta}_{av}$ satisfies

$$\dot{\tilde{\theta}}_{av} = \epsilon f_{av}(\tilde{\theta}_{av}) \tag{5.5.39}$$

where $f_{av}(\tilde{\theta})$ is defined as

$$f_{av}(\tilde{\theta}) = -\lim_{T\to\infty} \frac{1}{T} \int_{t_0}^{t_0+T} w_*(t)\, c_*^T \left[\int_0^t e^{A_*(t-\tau)} b_* w_*^T(\tau) d\tau \right] dt\, \tilde{\theta} \tag{5.5.40}$$

Note that $f_{av}(\tilde{\theta})$ is a linear function of $\tilde{\theta}$ so that the stability/instability of (5.5.38) for ϵ small is easily determined from the eigenvalues of the matrix in (5.5.36). As previously, the matrix in (5.5.40) may be written as the cross-correlation at 0 between $w_*(t)$ and

$$w_{*f}(t) := \int_0^t c_*^T e^{A_*(t-\tau)} b_* w_*(\tau)\, d\tau \tag{5.5.41}$$

Thus (5.5.40) may be written as

$$f_{av}(\tilde{\theta}) = -R_{w_* w_{*f}}(0)\, \tilde{\theta} \tag{5.5.42}$$

Frequency Domain Analysis

To derive a frequency domain interpretation, we assume that r is stationary. The spectral measure of w_* is related to that of r by

$$S_{w_*}(d\omega) = \hat{H}_{w_*r}^*(j\omega)\, \hat{H}_{w_*r}^T(j\omega)\, S_r(d\omega) \tag{5.5.43}$$

where the transfer function from r to w_* is $\hat{H}_{w_*r}(s)$. This transfer function is obtained by denoting the transfer function of the tuned plant

$$c_{0*}\, c_*^T(sI - A_*)^{-1} b_* = \hat{M}_*(s) \tag{5.5.44}$$

so that

$$\hat{H}_{w_*r}(s) = \begin{bmatrix} 1 \\ (sI-\Lambda)^{-1} b_\lambda \hat{P}^{-1} \hat{M}_* \\ \hat{M}_* \\ (sI-\Lambda)^{-1} b_\lambda \hat{M}_* \end{bmatrix} \tag{5.5.45}$$

The cross-correlation between w_* and w_{*f} is then given by

$$R_{w_* w_{*f}}(0) = \frac{1}{2\pi c_{0*}} \int_{-\infty}^{\infty} \hat{H}_{w_*r}^*(j\omega)\, \hat{H}_{w_*r}^T(j\omega)\, \hat{M}_*(j\omega)\, S_r(d\omega) \tag{5.5.46}$$

Note the similarity between (5.5.45), (5.5.46) and (4.5.6), (4.5.9) for averaging in the ideal case. The chief difference is the presence of the tuned plant transfer function $\hat{M}_*(s)$ in place of the model transfer function $\hat{M}(s)$. $\hat{M}_*(s)$ may not be strictly positive real, even if $\hat{M}(s)$ is so. Consequently $R_{w_* w_{*f}}(0)$ *may not be a positive semi-definite* matrix. Heuristically speaking, if a large part of the frequency support of the reference signal lies in a region where the real part of $\hat{M}_*(j\omega)$ is negative, then $R_{w_* w_{*f}}(0)$ may fail to be positive semi-definite.

It is easy to see that if all the eigenvalues of $R_{w_* w_{*f}}(0)$ are in the right half plane, the (simplified) overall system (5.5.38) is *globally* asymptotically stable for ϵ small enough. Also, if even one of the eigenvalues of $R_{w_* w_{*f}}(0)$ lies in the left half plane, then the system (5.5.38) is unstable (in the sense of Lyapunov). From the form of the integral in (5.5.46), one may deduce that a necessary condition for $R_{w_* w_{*f}}(0)$ to have no zero eigenvalues is for the reference input to have at least $2n$ points of support. In fact, heuristically speaking, for $R_{w_* w_{*f}}(0)$ to have no negative eigenvalues, the reference input is required to have at least $2n$ points of support in the frequency range where $\mathrm{Re}\,\hat{M}_*(j\omega) > 0$ (the reason that this is heuristic rather than precise is because the columns of $\hat{H}_{w_*r}(j\omega)$ may not be linearly independent at every set of $2n$ frequencies). Since the tuned plant transfer function $\hat{M}_*(s)$ is close to the model transfer function $\hat{M}(s)$ at least for low frequencies (where there are no unmodeled dynamics), it follows that to keep the adaptive system stable, sufficient excitation at lower frequencies is required. It is also important to see that the stability/instability criterion is both *signal-dependent* as well as dependent on the *tuned plant transfer function* $\hat{M}_*(s)$.

It is important at this point to note that all of the analysis is being performed on the averaged version of the simplified linearized system (5.5.38). As far as the original system (5.5.31) is concerned, we can make the following observations

a) If the simplified, linearized system (5.5.38) is exponentially stable, then the original system (5.5.37) is locally stable in the sense that if the tuned error and the initial conditions on $\tilde{e}_*$, $\tilde{\theta}$ are sufficiently small, trajectories will eventually be confined to a neighborhood of the origin (the size of the neighborhood depends on e_* and goes to zero as $|e_*(t)|$ goes to zero). The proof of this follows from theorem 5.3.1.

b) If the simplified linearized system is unstable, then the original system (5.5.37) is also unstable, using arguments from theorem 5.5.1.

Of course, the averaging analysis may be inconclusive if the averaged system $R_{w_* w_{*f}}(0)$ has some zero eigenvalues. In this instance, if $R_{w_* w_{*f}}(0)$ has at least one eigenvalue in the open left half plane, then the original system is unstable. However, if $R_{w_* w_{*f}}(0)$ has all its eigenvalues in the closed right half plane, including some at zero, the averaging is inconclusive for (5.5.38) and for (5.5.37). Simulations seem to suggest that, in this case, the parameter error vector $\tilde{\theta}$ driven by e_* drifts away from the origin in the presence of noise. This is what happens in Rohrs example (R3), where the reference input is only a DC input: e_* corresponding to the tuned error is small, since the closed loop plant matches the model at low frequencies, but its place is taken by the output disturbance which causes the parameters to drift away from their tuned values.

The result of this section also makes rigorous the heuristic explanation for the instability mechanism of the example in (R2) where a significant high frequency signal is present in a range where the tuned plant transfer function is not strictly positive real. A tuned plant is easily obtained by removing the unmodeled poles at $-15 \pm j2$ to get the tuned values $c_{0*} = 1.5$, $d_{0*} = 1$ identical to θ^*.

Example

In this section, we discuss an example from Riedle & Kokotovic [1985]. We consider the plant

$$\hat{P}(s) = \frac{k_p}{\mu s^2 + (1+\mu)s + 1} \qquad (5.5.47)$$

where $\mu > 0$ is a small parameter. The adaptive controller is designed assuming a first order plant with relative degree 1. Thus, we assume that the 'nominal' plant is of the form

$$P_{\theta^*} = \frac{k_p}{s+1} \qquad (5.5.48)$$

with k_p unknown. The model is of the form $1/(s+1)$ and we set the tuned value of c_0, namely c_{0*} to be $1/k_p$ for the analysis. For the example, k_p is chosen to be 1. The error system is

$$\begin{bmatrix} \dot{\tilde{e}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & \frac{-1}{\mu} & \frac{1}{\mu} r(t) \\ -g\, r(t) & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{e} \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -g \end{bmatrix} r(t)\, e_*(t) \qquad (5.5.49)$$

Note that (5.5.49) is simpler than (5.5.37) since there are no adaptive parameters in the feedback loop. $R_{w_* w_{*f}}(0)$ is a scalar, and is easily computed to be

$$R_{w_* w_{*f}}(0) = \int_{-\infty}^{\infty} \frac{1-\mu\omega^2}{(1-\mu\omega^2)^2 + (1+\mu)^2\omega^2} S_r(d\omega) \qquad (5.5.50)$$

Note that the integrand is positive for $|\omega| < 1/\sqrt{\mu}$ and negative for $|\omega| > 1/\sqrt{\mu}$. For example, if $\mu = 0.1$ and

$$r_1(t) = \sin 5t \qquad R_{w_* w_{*f}}(0) = -0.046$$

$$r_2(t) = 0.4\sin t + \sin 5t \qquad R_{w_* w_{*f}}(0) = 0.026$$

Thus, the first input results in an unstable system and the second in a stable one. These results are borne out in the simulations of Figure 5.15.

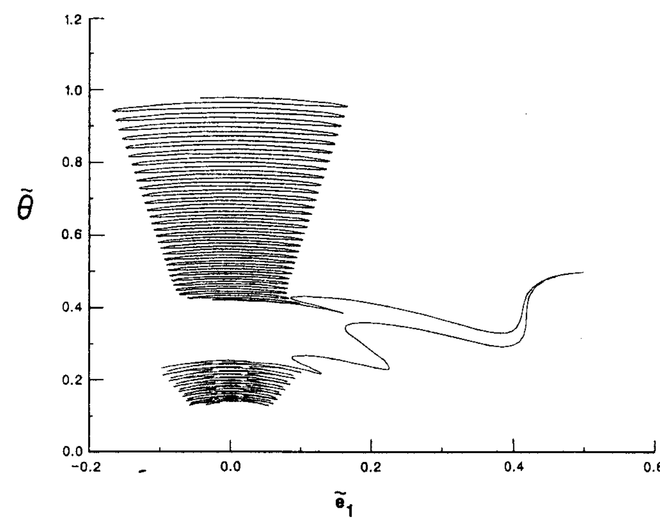

Figure 5.15 Stability–Instability Boundary for $r_1(t)$ and $r_2(t) = r_1(t) + 0.4\sin t$.

5.6 METHODS FOR IMPROVING ROBUSTNESS— QUALITATIVE DISCUSSION

Adaptive systems are not magically robust: several choices need to be carefully made in the course of design, and they need to explicitly take into account the limitations and flexibilities of the rather general algorithms presented in earlier chapters. We begin with a qualitative discussion of methods to improve the robustness of adaptive systems. A review of a few specific update law modifications is given in the next section.

5.6.1 Robust Identification Schemes

An important part of the adaptive control scheme is the identifier, or adaptation algorithm. When only parametric uncertainty is present, adaptive schemes are proved to be stable, with asymptotic tracking. Parameter convergence is not guaranteed in general, but is not necessary to achieve stability. In the presence of unmodeled dynamics and measurement noise, drift instabilities may occur, so that the spectral content of the input becomes important. The robustness of the identifier is fundamental to the robustness the adaptive system, and may be influenced by a careful design.

An initial choice of the designer is the frequency range of interest. In an adaptive control context, it is the frequency range over which accurate tracking is desired, and is usually limited by actuators' bandwith and sensor noise.

The order of the plant model must then be selected. The order should be sufficient to allow for modeling of the plant dynamics in the frequency range of interest. On the other hand, if the plant is of high order, a great deal of excitation (a number of independent frequencies) will be required. The presence of a large parameter vector in the identifier may also cause problems of numerical conditioning in the identification procedure. Then, the covariance matrix $R_w(0)$ (see Chapters 2 and 3) measuring the extent of persistent excitation, is liable to be ill-conditioned, resulting in slow parameter convergence along certain directions in parameter space. In summary, it is important to choose a low enough order plant model capable of representing all the plant dynamics in the frequency range of interest.

Filtering of the plant input and output signals is achieved by the observer, with a bandwith determined by the filter polynomial (denoted earlier $\hat{\lambda}(s)$). To reduce the effect of noise, it may be reasonable to further filter the regression vectors in the identification algorithm, so as to exclude the contribution of data from frequency regions lying outside the range of frequencies of importance to the controller (i.e. low pass filtering with a cut-off somewhat higher than the control bandwith).

The spectrum of the reference input is another parameter, partially left to the designer. Recall that the identifier identifies the portion of the plant dynamics in the frequency range of the input spectrum. Thus, it is important that the input signal: a) be rich enough to guarantee parameter convergence, and b) have energy content in the frequency range where the plant model is of sufficient order to represent the actual plant. The examples of Rohrs consisted of scenarios in which a) the input was not rich enough (only a DC signal), and b) the output had energy in the frequencies of the unmodeled dynamics (a DC signal and a high-frequency sinusoid). In the first case, noise caused parameter drift, consistent with a good low frequency model of the plant, into a region of instability. In the second, an incorrect plant model resulted in an unstable loop.

From a practical viewpoint, it is important to monitor the signal excitation in the identifier loop and to turn off the adaptation when the excitation is poor. This includes the case when the level of excitation is so low as to make it difficult to distinguish between the excitation and the noise. It is also clear that if the excitation is poor over periods of time where parameters vary, the parameter identification will be ineffectual. In such an event, the only cure is to inject extra perturbation signals into the reference input so as to provide excitation for the identification algorithm.

We summarize this discussion in the form of the following table for a robust identification scheme.

Steps of Robust Identification

	Step	Considerations
1.	Choice of the frequency range of interest	Frequency range over which tracking is desired
2.	Plant Order Determination	Modeling of the plant dynamics in the frequency range of interest Low
3.	Regressor Filter Selection	Filter high frequency components (unmodeled dynamics range)
4.	Reference Input Selection If not, check step 5	Sufficient richness Spectrum within frequency range of interest
5.	Turn off parameter update when not rich enough	

If the excitation is not rich over periods of time where parameters vary, add perturbation signal	Limit perturbation to plant

5.6.2 Specification of the Closed Loop Control Objective—Choice of the Reference Model and of Reference Input

The reference model must be chosen to reflect a desirable response of the closed-loop plant. From a robust control standpoint, however, control should only be attempted over a frequency range where a satisfactory plant model and controller parameterization exists. Therefore, the control objective (or reference model choice) should have a bandwidth no greater than that of the identifier. In particular, the reference model should not have large gain in those frequency regions in which the unmodeled dynamics are significant.

The choice of reference input is also one of the choices in the overall control objective. We indicated above how important the choice is for the identification algorithm. However, persistent excitation in the correct frequency range for identification may require added reference inputs not intended for tuned controller performance. In some applications (such as aircraft flight control), the insertion of perturbation signals into the reference input can result in undesirable dithering of the plant output. The reference input in adaptive systems plays a dual role, since the input is required both for generating the reference output required for tracking, as well as furnishing the excitation needed for parameter convergence (this dual role is sometimes referred to as the *dual control concept*).

5.6.3 The Usage of Prior Information

The schemes and stability proofs thus far have involved very little *a priori* knowledge about the plant under control. In practice, one is often confronted with systems which are fairly well modeled, except for a few unknown and uncertain components which need to be identified. In order to use the schemes in the form presented so far, all of the prior knowledge needs to be completely discounted. This, however, increases the order of complexity of the controller, resulting in extra requirements on the amount of excitation needed. In certain instances, the problem of incorporating prior information can be solved in a neat and consistent fashion—for this we refer the reader to Section 6.1 in the next chapter.

5.6.4 Time Variation of the Parameters

The adaptive control algorithms so far have been derived and analyzed for the case of unknown but fixed parameter values. In practice, adaptive control is most useful in scenarios involving slowly changing plant parameters. In these instances the estimator needs to converge much faster than the rate of plant parameter variation. Further, the estimator needs to discount old input-output data: old data should be discounted quickly enough to allow for the estimator parameters to track the time-varying ones. The discounting should not, however, be too fast since this would involve an inconsistency of parameter values and sensitivity to noise.

We conclude this section with Figure 5.16, inspired from Johnson [1988], indicating the desired ranges of the different dynamics in an adaptive system.

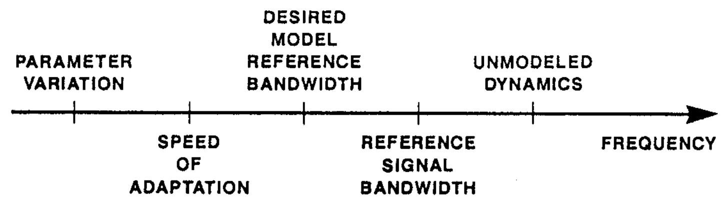

Figure 5.16 Desirable Bandwidths of Operation of an Adaptive Control System.

5.7 ROBUSTNESS VIA UPDATE LAW MODIFICATIONS

In the previous sections, we reviewed some of the reasons for the loss of robustness in adaptive schemes and qualitatively discussed how to remedy them. In this section, we present modifications of the parameter update laws which were recently proposed as robustness enhancement techniques.

5.7.1 Deadzone and Relative Deadzone

The general idea of a *deadzone* is to stop updating the parameters when the excitation is insufficient to distinguish between the regressor signal and the noise. Thus, the adaptation is turned off when the identifier error is smaller than some threshold.

More specifically, consider the input error direct adaptive control algorithm with the generalized gradient algorithm and projection. The update law with deadzone is given by

$$\dot{\theta} = -g \frac{e_2 v}{1 + \gamma v^T v} \qquad \text{if } |e_2| > \Delta \qquad (5.7.1)$$

$$\dot{\theta} = 0 \qquad \text{if } |e_2| \leq \Delta \qquad (5.7.2)$$

and as before, if $c_0 = c_{min}$ and $\dot{c}_0 < 0$, then set $\dot{c}_0 = 0$. The parameter Δ in equations (5.7.1) and (5.7.2) represents the size of the deadzone. Similarly, the output error direct adaptive control algorithm with gradient algorithm is modified to

$$\dot{\bar{\theta}} = -g e_1 \bar{v} \qquad \text{if } |e_1| > \Delta \qquad (5.7.3)$$

$$\dot{\bar{\theta}} = 0 \qquad \text{if } |e_1| \leq \Delta \qquad (5.7.4)$$

where Δ is, as before, the deadzone threshold. It is easy to see how the other schemes (including the least-squares update laws) are modified.

The most critical part in the application of these schemes is the selection of the width of the deadzone Δ. If the deadzone Δ is too large, e_2 in equations (5.7.1), (5.7.2) and e_1 in (5.7.3), (5.7.4) will not tend to zero, but will only be asymptotically bounded by a large Δ, resulting in undesirable closed-loop performance. A number of recent papers (for example, Peterson & Narendra [1982], Samson [1983], Praly [1983], Sastry [1984], Ortega, Praly & Landau [1985], Kreisselmeier & Anderson [1986], Narendra & Annaswamy [1986]) have suggested different techniques for the choice of the deadzone Δ. The approach taken by Peterson & Narendra [1982]—for the case when the plant output is corrupted by additive noise—and by Praly [1983] and Sastry [1984]—for the case of both output noise and unmodeled dynamics—is to use some prior bounds on the disturbance magnitude and some prior knowledge about the plant to find a (conservative) bound on Δ and establish that the tracking error eventually converges to the region $|e_1| \leq \Delta$. The bounds on Δ which follow from their calculations are, however, extremely conservative. From a practical standpoint, these results are to be interpreted as mere existence results. Practically, one would choose Δ from observing the noise floor of the parameter update variable e_1 (with no exogenous reference input present). It is also possible to modify it on-line depending on the quality of the data.

The approach of Samson [1983], Ortega, Praly & Landau [1985] and Kreisselmeier & Anderson [1986] is somewhat different in that it involves a deadzone size Δ which is not determined by e_1 or e_2 alone, but by how large the regressor signal in the adaptive loop is (the deadzone acts on a suitably normalized, relative identification error). The logic behind this so-called *relative deadzone* is that if the regressor vector is large, then the identification error may be large even for a small transfer function error due to unmodeled dynamics. The details of the relative deadzone are somewhat involved (the three papers referenced above are also for discrete time algorithms). However, the adaptive law can only guarantee that the *relative* (or normalized) identification error becomes smaller than the deadzone eventually. Thus, if the closed-loop system were unstable, the absolute identification error could be unbounded. To complete the proofs of stability with relative deadzones, it is then important to prove that the regressor vector is bounded. It is claimed (cf. Kreisselmeier & Anderson [1986]) that the relative deadzone approach will not suffer from "bursting," unlike the absolute deadzone approach.

5.7.2 Leakage Term (σ–Modification)

Ioannou & Kokotovic [1983] suggested modifying the parameter update law to counteract the drift of parameter values into regions of instability in the absence of persistent excitation. The original form of the modification is, for the direct output error scheme

$$\dot{\bar{\theta}} = -g e_1 \bar{v} - \sigma \bar{\theta} \qquad (5.7.5)$$

where σ is chosen small but positive to keep $\bar{\theta}$ from growing unbounded. Two other interesting modifications in the spirit of (5.7.5) are

$$\dot{\bar{\theta}} = -g e_1 \bar{v} - \sigma(\bar{\theta} - \bar{\theta}_0) \qquad (5.7.6)$$

where $\bar{\theta}_0$ is a prior estimate of $\bar{\theta}$ (for this and other modifications see Ioannou [1986] and Ioannou and Tsakalis [1986]), and one suggested by Narendra and Annaswamy [1987]

$$\dot{\bar{\theta}} = -g e_1 \bar{v} - \sigma |e_1| \bar{\theta} \qquad (5.7.7)$$

Both (5.7.6) and (5.7.7) attempt to capture the spirit of (5.7.5) without its drawback of causing $\bar{\theta} \to 0$ if e_1 is small. Equation (5.7.6) tries to bias the direction of the drift towards $\bar{\theta}_0$ rather than 0 and (5.7.7) tries to turn off the drift towards 0 when $|e_1|$ is small. The chief advantage of the update law (5.7.7) is that it retains features of the algorithm without leakage (such as convergence of the parameters to their true values when the excitation is persistent). Also, the algorithm (5.7.7) may be less susceptible to bursting than (5.7.5), though this claim has not been fully substantiated.

5.7.3 Regressor Vector Filtering

The concept of low pass filtering or pre-conditioning the regressor vector in the parameter update law was discussed in Section 5.6. It is usually accomplished by low pass filtering of the process input and output in the identification algorithm and is widely prevalent (cf. the remarks in Wittenmark & Astrom [1984]). Some formalization of the concept and its analysis is in the work of Johnson, Anderson & Bitmead [1984]. The logic is that low pass filtering tends to remove noise and contributions of high frequency unmodeled dynamics.

5.7.4 Slow Adaptation, Averaging and Hybrid Update Laws

A key characteristic of the parameter update laws even with the addition of deadzones, leakage and regressor vector filtering is their "impatience." Thus, if the identification error momentarily becomes large, perhaps for reasons of spurious, transient noise, the parameter update operates instantaneously. A possible cure for this impatience is to slow down the adaptation. In Chapter 4, we studied in great detail the averaging effects of using a small adaptation gain on the parameter trajectories. In fact, a reduction of the effect of additive noise (by averaging) is also observed.

Another modification of the parameter update law in the same spirit is the so-called hybrid update law involving discrete updates of continuous time schemes. One such modification of the gradient update law (due to Narendra, Khalifa & Annaswamy [1985]) is

$$\theta(t_{k+1}) = \theta(t_k) - \int_{t_k}^{t_{k+1}} g\, e_1 v\, dt \tag{5.7.8}$$

In (5.7.8), the t_k refer to parameter update times, and the controller parameters are held constant on $[t_k, t_{k+1}]$. The law (5.7.8) relies on the averaging inherent in the integral to remove noise.

Slow adaptation and hybrid adaptation laws suffer from two drawbacks. First, they result in undesirable transient behavior if the initial parameter estimates result in an unstable closed loop (since stabilization is slow). Second, they are incapable of tracking fast parameter variations. Consequently, the best way to use them is after the initial part of the transient in the adaptation algorithm or a short while after a parameter change, which the "impatient" algorithms are better equipped to handle.

5.8 CONCLUSIONS

In this chapter, we studied the problem of the robustness of adaptive systems, that is, their ability to maintain stability despite modeling errors and measurement noise.

We first reviewed the Rohrs examples, illustrating several mechanisms of instability. Then, we derived a general result relating exponential stability to robustness. The result indicated that the property of

exponential stability is robust, while examples show that the BIBS stability property is not (that is, BIBS stable systems can become unstable in the presence of arbitrarily small disturbances). In practice, the amplitude of the disturbances should be checked against robustness margins to determine if stability is guaranteed. The complexity of the relationship between the robustness margins and known parameters, and the dependence of these margins on external signals unfortunately made the result more conceptual than practical.

The mechanisms of instability found in the Rohrs examples were discussed in view of the relationship between exponential stability and robustness, and a heuristic analysis gave additional insight. Further explanations of the mechanisms of instability were presented, using an averaging analysis. Finally, various methods to improve robustness were reviewed, together with recently proposed update law modifications.

We have attempted to sketch a sampling of what is a very new and active area of research in adaptive systems. We did not give a formal statement of the convergence results for all the adaptation law modifications. The results are not yet in final form in the literature and estimates accruing from systematic calculations are conservative and not very insightful. A great deal of the preceding discussion should serve as design guidelines: the exact design trade-offs will vary from application to application. The general message is that it is perhaps not a good idea to treat adaptive control design as a "black box" problem, but rather to use as much process knowledge as is available in a given application.

A guideline for design might run as follows

a) Determine the frequency range beyond which one chooses not to model the plant (where unmodeled dynamics appear) and find a parameterization which is likely to yield a good model of the plant in this frequency range, yet without excessive parameterization. If prior information is available, use it (see Section 6.1 for more on this).

b) Choose a reference model (performance objective) whose bandwidth does not extend into the range of unmodeled dynamics.

c) In the course of adaptation, implement the adaptive law with a deadzone whose size is determined by observing the amount of noise in the absence of exogenous input. Also, *monitor the excitation* and turn off the adaptation when the excitation is not rich over a large interval of time. If necessary, *inject extra excitation* into the exogenous reference input as a perturbation signal. If it appears that the plant parameters are not varying very rapidly, slow down the rate of adaptation or use a hybrid update algorithm (this is rather like a variable time step feature in numerical integration routines). Other modifications, such as leakage, may be added as desired.

d) Implement the appropriate start-up features for the algorithm using prior knowledge about the plant to choose initial parameter values and include "safety nets" to cover start-up, shut-down and transitioning between various modes of operation of the overall controller.

The guidelines given in this chapter are for the most part conceptual: in applications, questions of numerical conditioning of signals, sampling intervals (for digital implementations), anti-aliasing filters (for digital implementations), controller-architecture featuring several levels of interruptability, resetting, and so on are important. Even with a considerable wealth of theory and analysis of the algorithms, the difference an adaptive controller makes in a given application is chiefly due to the art of the designer!

CHAPTER 6

ADVANCED TOPICS IN IDENTIFICATION AND ADAPTIVE CONTROL

6.1 USE OF PRIOR INFORMATION

6.1.1 Identification of Partially Known Systems

We consider in this section the problem of identifying partially known single-input single-output (SISO) transfer functions of the form

$$\hat{P}(s) = \frac{\hat{N}_0(s) + \sum_{i=1}^{m} \alpha_i \hat{N}_i(s)}{\hat{D}_0(s) - \sum_{j=1}^{n} \beta_j \hat{D}_j(s)} \qquad (6.1.1)$$

where $\hat{N}_i$ and $\hat{D}_j$ are known, proper, stable rational transfer functions and α_i, β_j are unknown, real parameters. The identification problem is to identify α_i, β_j from input-output measurements of the system. The problem was recently addressed by Clary [1984], Dasgupta [1984], and Bai and Sastry [1986].

The representation (6.1.1) is general enough to model several kinds of "partially known" systems.

Examples

a) *Network functions of RLC circuits with some elements unknown.* Consider for example the circuit of Figure 6.1, with the resistor R unknown (the circuit is drawn as a two port to exhibit the unknown